Second Order Backward Stochastic Differential Equations with Quadratic Growth*

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Abstract

We extend the wellposedness results for second order backward stochastic differential equations introduced by Soner, Touzi and Zhang [24] to the case of a bounded terminal condition and a generator with quadratic growth in the z variable. More precisely, we obtain uniqueness through a representation of the solution inspired by stochastic control theory, and we obtain two existence results using two different methods. In particular, we obtain the existence of the simplest purely quadratic 2BSDEs through the classical exponential change, which allows us to introduce a quasi-sure version of the entropic risk measure. As application, we also study robust risk-sensitive control problems. Finally, we prove a Feynman-Kac formula and a probabilistic representation for fully nonlinear PDEs in this setting.

Key words: Second order backward stochastic differential equation, quadratic growth, r.c.p.d., Feyman-Kac, fully nonlinear PDEs, quasi-sure.

AMS 2000 subject classifications: 60H10, 60H30

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1 Introduction

Backward stochastic differential equations (BSDEs for short) appeared for the first time in Bismut [3] in the linear case. However, they only became a popular field of research after the seminal paper of Pardoux and Peng [20], mainly because of the very large scope of their domain of applications, ranging from stochastic control to mathematical finance.

On a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ generated by an $\mathbb{R}^d$-valued Brownian motion B, solving a BSDE amounts to finding a pair of progressively measurable processes (Y, Z) such that

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s,\ t \in [0, T],\ \mathbb{P} - a.s.$$

where f (called the generator) is a progressively measurable function and ξ is an $\mathcal{F}_T$-measurable random variable.

Pardoux and Peng [20] proved existence and uniqueness of the above BSDE provided that the function f is uniformly Lipschitz in y and z and that ξ and $f_s(0, 0)$ are square integrable. Then, in [21], they proved that if the randomness in f and ξ is induced by the current value of a state process defined by a forward stochastic differential equation, then the solution to the BSDE could be linked to the solution of a semilinear PDE by means of a generalized Feynman-Kac formula. This link then opened the way to probabilistic numerical methods for solving semilinear PDEs, which particularly well suited for highly dimensional problems (see Bouchard and Touzi [4] among many other).

Nonetheless, the class of fully nonlinear PDEs remained inaccessible when considering only BSDEs, until a recent work of Cheredito, Soner, Touzi and Victoir [9]. They introduced a notion of second order BSDEs (2BSDEs), which were then proved to be naturally linked to fully nonlinear PDEs. However, only a uniqueness result in the Markovian case was proved, and no existence result (apart from trivial ones) were available. Following this work, Soner, Touzi and Zhang [24] gave a new definition of 2BSDEs, and thus provided a complete theory of existence and uniqueness under uniform Lipschitz conditions similar to those of Pardoux and Peng. Their key idea was to reinforce the condition that the 2BSDE must hold $\mathbb{P} - a.s.$ for every probability measure $\mathbb{P}$ in a non-dominated class of mutally singular measures (see Section 2 for precise definitions). Let us describe briefly the intuition behind their definition.

Suppose that we want to study the following fully non-linear PDE

$$-\frac{\partial u}{\partial t} - h\left(t, x, u(t, x), Du(t, x), D^2u(t, x)\right) = 0, \qquad u(T, x) = g(x). \tag{1.1}$$

If the function $\gamma \mapsto h(t, x, r, p, \gamma)$ is assumed to be convex, then it is equal to its double Fenchel-Legendre transform, and if we denote its Fenchel-Legendre transform by f, we have

$$h(t, r, p, \gamma) = \sup_{a\geq 0} \left\{\frac{1}{2}a\gamma - f(t, x, r, p, a)\right\} \tag{1.2}$$

Then, from (1.2), we expect, at least formally, that the solution u of (1.1) is going to verify

$$u(t,x) = \sup_{a\geq 0} u^a(t,x),$$

where u^a is defined as the solution of the following semi-linear PDE

$$-\frac{\partial u^a}{\partial t} - \frac{1}{2}aD^2u^a(t,x) + f\left(t,x,u^a(t,x),Du^a(t,x),a\right) = 0, \qquad u^a(T,x) = g(x). \tag{1.3}$$

Since u^a is linked to a classical BSDE, the 2BSDE associated to u should correspond (in some sense) to the supremum of the family of BSDEs indexed by a. Furthermore, changing the process a can be achieved by changing the probability measure under which the BSDE is written. However, this amounts to changing the quadratic variation of the martingale driving the BSDE, and therefore leads to a family of mutually singular probability measures. In these respects, the 2BSDE theory shares deep links with the theory of quasi-sure stochastic analysis of Denis and Martini [10] and the theory of G-expectation of Peng [22].

Following the breakthrough of [24], Possamai [23] extended the existence and uniqueness result for 2BSDEs to the case of a generator having linear growth and satisfying a monotonicity condition. Motivated by a robust utility maximization problem under volatility uncertainty (see the accompanying paper [19]), our aim here is to go beyond the results of [23] to prove an existence and uniqueness result for 2BSDEs whose generator has quadratic growth in z.

The question of existence and uniqueness of solutions to these quadratic equations in the classical case was first examined by Kobylanski [17], who proved existence and uniqueness of a solution by means of approximation techniques borrowed from the PDE litterature, when the generator is continuous and has quadratic growth in z and the terminal condition ξ is bounded. Then, Tevzadze [28] has given a direct proof for the existence and uniqueness of a bounded solution in the Lipschitz-quadratic case, proving the convergence of the usual Picard iteration. Following those works, Briand and Hu [6] have extended the existence result to unbounded terminal condition with exponential moments and proved uniqueness for a convex coefficient [7]. Finally, Barrieu and El Karoui [2] recently adopted a completely different approach, embracing a forward point of view to prove existence under conditions similar to those of Briand and Hu.

In this paper, we propose two very different methods to prove the wellposedness in the 2BSDE case. First, we recall some notations in Section 2 and prove a uniqueness result in Section 3 by means of a priori estimates and a representation of the solution inspired by the stochastic control theory. Then, Section 4 is devoted to the study of approximation techniques for the problem of existence of a solution. We advocate that since we are working under a family of non-dominated probability measures, the monotone or dominated convergence theorem may fail. This is a major problem, and we spend some time explaining why, in general, the classical methods using exponential changes fail for 2BSDEs. Nonetheless, using very recent results of Briand and Elie [8], we are able to show a first existence result using an approximation method. Then in Section 5, we use a completely different method introduced by Soner, Touzi and Zhang [24] to construct the solution to

the quadratic 2BSDE path by path. Next, we use these results in Section 6 to study an application of 2BSDEs with quadratic growth to robust risk-sensitive control problems. Finally, in Section 7, we extend the results of Soner, Touzi and Zhang [24] on the connections between fully non-linear PDEs and 2BSDEs to the quadratic case.

2 Preliminaries

Let $\Omega := \left\{\omega \in C([0,T],\mathbb{R}^d) : \omega_0 = 0\right\}$ be the canonical space equipped with the uniform norm $\|\omega\|_\infty := \sup_{0\leq t\leq T}|\omega_t|$, B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0\leq t\leq T}$ the filtration generated by B, and $\mathbb{F}^+ := \left\{\mathcal{F}_t^+\right\}_{0\leq t\leq T}$ the right limit of $\mathbb{F}$.

2.1 A first set of probability measures

Our aim here is to give a correct mathematical basis to the intuitions we provided in the Introduction. We first recall that by the results of Karandikar [15], we can give pathwise definitions of the quadratic variation $\langle B\rangle_t$ and its density $\widehat{a}_t$.

Let $\overline{\mathcal{P}}_W$ denote the set of all local martingale measures $\mathbb{P}$ (i.e. the probability measures $\mathbb{P}$ under which B is a local martingale) such that

$$\langle B\rangle_t \text{ is absolutely continuous in } t \text{ and } \widehat{a} \text{ takes values in } \mathbb{S}_d^{>0},\ \mathbb{P}-a.s. \tag{2.1}$$

where $\mathbb{S}_d^{>0}$ denotes the space of all $d\times d$ real valued positive definite matrices.).

As in [24], we concentrate on the subclass $\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_W$ consisting of all probability measures

$$\mathbb{P}^\alpha := \mathbb{P}_0\circ(X^\alpha)^{-1} \text{ where } X_t^\alpha := \int_0^t \alpha_s^{1/2}dB_s,\ t\in[0,T],\ \mathbb{P}_0-a.s. \tag{2.2}$$

for some $\mathbb{F}$-progressively measurable process α satisfying $\int_0^T|\alpha_s|\,ds < +\infty$. We recall from [25] that every $\mathbb{P}\in\overline{\mathcal{P}}_S$ satisfies the Blumenthal zero-one law and the martingale representation property.

Notice that the set $\overline{\mathcal{P}}_S$ is bigger than the set $\widetilde{\mathcal{P}}_S$ introduced in [23], which is defined by

$$\widetilde{\mathcal{P}}_S := \left\{\mathbb{P}^\alpha \in \overline{\mathcal{P}}_S,\ \underline{a}\leq\alpha\leq\bar{a},\ \mathbb{P}_0-a.s.\right\}, \tag{2.3}$$

for fixed matrices $\underline{a}$ and $\bar{a}$ in $\mathbb{S}_d^{>0}$.

2.2 The Generator and the final set $\mathcal{P}_H$

Before defining the spaces under which we will be working or defining the 2BSDE itself, we first need to restrict one more time our set of probability measures, using explicitely the generator of the 2BSDE.

Following the PDE intuition recalled in the Introduction, let us first consider a map $H_t(\omega,y,z,\gamma) : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times D_H \to \mathbb{R}$, where $D_H\subset\mathbb{R}^{d\times d}$ is a given subset

containing 0. As expected, we define its Fenchel-Legendre conjugate w.r.t.γ by

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2} \mathrm{Tr}(a\gamma) - H_t(\omega, y, z, \gamma) \right\} \text{ for } a \in \mathbb{S}_d^{>0}$$

$$\widehat{F}_t(y, z) := F_t(y, z, \widehat{a}_t) \text{ and } \widehat{F}_t^0 := \widehat{F}_t(0, 0).$$

We denote by $D_{F_t(y,z)}$ the domain of F in a for a fixed (t, ω, y, z), and as in [24] we restrict the probability measures in $\mathcal{P}_H \subset \overline{\mathcal{P}}_S$

Definition 2.1. *$\mathcal{P}_H$ consists of all $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that*

$$\underline{a}_{\mathbb{P}} \leq \widehat{a} \leq \bar{a}_{\mathbb{P}}, \; dt \times d\mathbb{P} - a.s. \text{ for some } \underline{a}_{\mathbb{P}}, \bar{a}_{\mathbb{P}} \in \mathbb{S}_d^{>0}, \text{ and } \widehat{a}_t \in D_{F_t(y,z)}.$$

Remark 2.1. *The restriction to the set $\mathcal{P}_H$ obeys two imperatives. First, since $\widehat{F}$ is destined to be the generator of our 2BSDE, we obviously need to restrict ourselves to probability measures such that $\widehat{a}_t \in D_{F_t(y,z)}$. Moreover, we also restrict the measures considered to the ones such that the density of the quadratic variation of B is bounded to ensure that B is actually a true martingale under each of those probability measures. This will be important to obtain a priori estimates.*

2.3 Spaces of interest

We now provide the definition of the spaces which will be used throughout the paper.

2.3.1 Quasi-sure spaces

$\mathbb{L}_H^\infty$ denotes the space of all $\mathcal{F}_T$-measurable scalar r.v. ξ with

$$\|\xi\|_{\mathbb{L}_H^\infty} := \sup_{\mathbb{P} \in \mathcal{P}_H} \|\xi\|_{L^\infty(\mathbb{P})} < +\infty.$$

$\mathbb{H}_H^p$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}^d$-valued processes Z with

$$\|Z\|_{\mathbb{H}_H^p}^p := \sup_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |\widehat{a}_t^{1/2} Z_t|^2 dt \right)^{\frac{p}{2}} \right] < +\infty.$$

$\mathbb{D}_H^\infty$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes Y with

$$\mathcal{P}_H - q.s. \text{ càdlàg paths, and } \|Y\|_{\mathbb{D}_H^\infty} := \sup_{0 \leq t \leq T} \|Y_t\|_{L_H^\infty} < +\infty.$$

Finally, we denote by $\mathrm{UC}_b(\Omega)$ the collection of all bounded and uniformly continuous maps $\xi : \Omega \to \mathbb{R}$ with respect to the $\|\cdot\|_\infty$-norm, and we let

$$\mathcal{L}_H^\infty := \text{the closure of } \mathrm{UC}_b(\Omega) \text{ under the norm } \|\cdot\|_{\mathbb{L}_H^\infty}.$$

Remark 2.2. *We emphasize that all the above norms and spaces are the natural generalizations of the standard ones when working under a single probability measure.*

2.3.2 The space $\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$ and important properties

It is a well known fact that the Z component of the solution of a quadratic BSDE with a bounded terminal condition belongs to the so-called BMO space. Since this link will be extended and used intensively throughout the paper, we will recall some results and definitions for the BMO space, and then extend them to our quasi-sure framework. We first recall (with a slight abuse of notation) the definition of the BMO space for a given probability measure $\mathbb{P}$.

Definition 2.2. $\mathrm{BMO}(\mathbb{P})$ *denotes the space of all* $\mathbb{F}^+$*-progressively measurable* $\mathbb{R}^d$*-valued processes* Z *with*

$$\|Z\|_{\mathrm{BMO}(\mathbb{P})} := \sup_{\tau\in\mathcal{T}_0^T} \left\|\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T |\widehat{a}_t^{1/2} Z_t|^2 dt\right]\right\|_\infty < +\infty,$$

where $\mathcal{T}_0^T$ *is the set of* $\mathcal{F}_t$ *stopping times taking their values in* $[0,T]$.

We also recall the so called energy inequalities (see [16] and the references therein). Let $Z \in \mathrm{BMO}(\mathbb{P})$ and $p \geq 1$. Then we have

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{a}_s^{1/2} Z_s\right|^2 ds\right)^p\right] \leq 2p! \left(4 \left\|Z\right\|^2_{\mathbb{H}^2_{BMO}}\right)^p. \tag{2.4}$$

The extension to a quasi-sure framework is then naturally given by the following space.

$\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}^d$-valued processes Z with

$$\|Z\|_{\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})} := \sup_{\mathbb{P}\in\mathcal{P}_H} \|Z\|_{\mathrm{BMO}(\mathbb{P})} < +\infty.$$

The main interest of the BMO spaces is that if a process Z belongs to it, then the stochastic integral $\int_0^\cdot Z_s dB_s$ is a uniformly integrable martingale, which in turn allows us to use it for changing the probability measure considered via Girsanov's Theorem. The two following results give more detailed results in terms of L^r integrability of the corresponding Doléans-Dade exponentials.

Lemma 2.1. *Let* $Z \in \mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$*. Then there exists* $r > 1$*, such that*

$$\sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\left(\mathcal{E}\left(\int_0^\cdot Z_s dB_s\right)\right)^r\right] < +\infty.$$

Proof. By Theorem 3.1 in [16], we know that if $\|Z\|_{\mathrm{BMO}(\mathbb{P})} \leq \Phi(r)$ for some one-to-one function Φ from $(1,+\infty)$ to $\mathbb{R}_+^*$, then $\mathcal{E}\left(\int_0^\cdot Z_s dB_s\right)$ is in $L^r(\mathbb{P})$. Here, since $Z \in \mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$, the same r can be used for all the probability measures. □

Lemma 2.2. *Let* $Z \in \mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$*. Then there exists* $r > 1$*, such that for all* $t \in [0,T]$

$$\sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}_t^{\mathbb{P}}\left[\left(\frac{\mathcal{E}\left(\int_0^t Z_s dB_s\right)}{\mathcal{E}\left(\int_0^T Z_s dB_s\right)}\right)^{\frac{1}{r-1}}\right] < +\infty.$$

Proof. This is a direct application of Theorem 2.4 in [16] for all $\mathbb{P} \in \mathcal{P}_H$. $\square$

We emphasize that the two previous Lemmas are absolutely crucial to our proof of uniqueness and existence. Besides, they also play a major role in our accompanying paper [19].

2.4 The definition of the 2BSDE

Everything is now ready to define the solution of a 2BSDE. We shall consider the following 2BSDE

$$Y_t = \xi + \int_t^T \widehat{F}_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t,\ 0 \leq t \leq T,\ \mathcal{P}_H - q.s. \tag{2.5}$$

Definition 2.3. *We say $(Y, Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ is a solution to 2BSDE (2.5) if*

- $Y_T = \xi$, $\mathcal{P}_H - q.s.$
- *For all $\mathbb{P} \in \mathcal{P}_H$, the process $K^\mathbb{P}$ defined below has non-decreasing paths $\mathbb{P} - a.s.$*
$$K^\mathbb{P}_t := Y_0 - Y_t - \int_0^t \widehat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ 0 \leq t \leq T,\ \mathbb{P} - a.s. \tag{2.6}$$
- *The family $\left\{K^\mathbb{P}, \mathbb{P} \in \mathcal{P}_H\right\}$ satisfies the minimum condition*
$$K^\mathbb{P}_t = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,inf}^\mathbb{P}} \mathbb{E}^{\mathbb{P}'}_t \left[K^{\mathbb{P}'}_T\right],\ 0 \leq t \leq T,\ \mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_H, \tag{2.7}$$
where
$$\mathcal{P}_H(t^+, \mathbb{P}) := \left\{\mathbb{P}' \in \mathcal{P}_H,\ \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_{t^+}\right\}.$$

Moreover if the family $\left\{K^\mathbb{P}, \mathbb{P} \in \mathcal{P}_H\right\}$ can be aggregated into a universal process K, we call (Y, Z, K) a solution of 2BSDE (2.5).

Remark 2.3. *Let us comment on this definition. As already explained, the PDE intuition leads us to think that the solution of a 2BSDE should be a supremum of solution of standard BSDEs. Therefore for each $\mathbb{P}$, the role of the non-decreasing process $K^\mathbb{P}$ is in some sense to "push" the process Y to remain above the solution of the BSDE with terminal condition ξ and generator $\widehat{F}$ under $\mathbb{P}$. In this regard, 2BSDEs share some similarities with reflected BSDEs.*

Pursuing this analogy, the minimum condition (2.7) tells us that the processes $K^\mathbb{P}$ act in a "minimal" way (exactly as implied by the Skorohod condition for reflected BSDEs), and we will see in the next Section that it implies uniqueness of the solution. Besides, if the set $\mathcal{P}_H$ was reduced to a singleton $\{\mathbb{P}\}$, then (2.7) would imply that $K^\mathbb{P}$ is a martingale and a non-decreasing process and is therefore null. Thus we recover the standard BSDE theory.

Finally, we would like to emphasize that in the language of G-expectation of Peng [22], (2.7) is equivalent, at least if the family can be aggregated into a process K, to saying that $-K$ is a G-martingale. This link has already observed in [26] where the authors proved the G-martingale representation property, which formally corresponds to a 2BSDE with a generator equal to 0.

2.5 Assumptions

We finish this Section by giving the properties which will be assumed to be satisfied by the generator $\widehat{F}$ of the 2BSDE.

Assumption 2.1. (i) *The domain $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z).*

(ii) *For fixed (y, z, γ), F is $\mathbb{F}$-progressively measurable.*

(iii) *F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.*

(iv) *F is continuous in z and has the following growth property. There exists $(\alpha, \beta, \gamma) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^*$ such that*

$$|F_t(\omega, y, z, a)| \leq \alpha + \beta |y| + \frac{\gamma}{2}\left|a^{1/2}z\right|^2, \text{ for all } (t, y, z, \omega, a).$$

(v) *F is C^1 in y and C^2 in z, and there are constants r and θ such that for all (t, ω, y, z, a),*

$$|D_y F_t(\omega, y, z, a)| \leq r, \; |D_z F_t(\omega, y, z, a)| \leq r + \theta \left|a^{1/2}z\right|,$$

$$|D^2_{zz} F_t(\omega, y, z, a)| \leq \theta.$$

Remark 2.4. *Let us comment on the above assumptions. Assumptions 2.1* (i) *and* (iii) *are taken from [24] and are needed to deal with the technicalities induced by the quasi-sure framework. Assumptions 2.1* (ii) *and* (iv) *are quite standard in the classical BSDE litterature. Finally, Assumption 2.1* (v) *was introduced by Tevzadze in [28] for quadratic BSDEs. It allowed him to prove existence of quadratic BSDEs through fixed point arguments. This is this consequence which will be used for technical reasons in Section 5.*

However, it was also proved in [28], that if both the terminal condition and $\widehat{F}^0$ were small enough, then Assumption 2.1 (v) *can be replaced by a weaker one. We will therefore sometimes consider*

Assumption 2.2. *Let points* (i) *to* (iv) *of Assumption 2.1 hold and*

(v) *We have the following "local Lipschitz" assumption in z, $\exists \mu > 0$ and a progressively measurable process $\phi \in \mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$ such that for all (t, y, z, z', ω, a),*

$$\left|F_t(\omega, y, z, a) - F_t(\omega, y, z', a) - \phi_t.a^{1/2}(z - z')\right| \leq \mu a^{1/2} |z - z'| \left(\left|a^{1/2}z\right| + \left|a^{1/2}z'\right|\right).$$

(vi) *We have the following uniform Lipschitz-type property in y*

$$\left|F_t(\omega, y, z, a) - F_t(\omega, y', z, a)\right| \leq C \left|y - y'\right|, \text{ for all } (y, y', z, t, \omega, a).$$

Furthermore, we observe that our subsequent proof of uniqueness of a solution of our quadratic 2BSDE only uses Assumption (2.2).

Remark 2.5. *By Assumption 2.1*(v)*, we have that $\widehat{F}^0_t$ is actually bounded, so the strong integrability condition*

$$\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \left|\widehat{F}^0_t\right|^{\kappa} dt\right)^{\frac{2}{\kappa}}\right] < +\infty,$$

with a constant $\kappa \in (1,2]$ introduced in [24] is not needed here. Moreover, this implies that $\widehat{a}$ always belongs to D_{F_t} and thus that $\mathcal{P}_H$ is not empty.

3 A priori estimates and uniqueness of the solution

Before proving some a priori estimates for the solution of the 2BSDE (2.5), we will first prove rigorously the intuition given in the Introduction saying that the solution of the 2BSDE should be, in some sense, a supremum of solution of standard BSDEs. Hence, for any $\mathbb{P} \in \mathcal{P}_H$, $\mathbb{F}$-stopping time τ, and $\mathcal{F}_\tau$-measurable random variable $\xi \in \mathbb{L}^\infty(\mathbb{P})$, we define $(y^{\mathbb{P}}, z^{\mathbb{P}}) := (y^{\mathbb{P}}(\tau,\xi), z^{\mathbb{P}}(\tau,\xi))$ as the unique solution of the following standard BSDE (existence and uniqueness have been proved under our assumptions by Tevzadze in [28])

$$y^{\mathbb{P}}_t = \xi + \int_t^\tau \widehat{F}_s(y^{\mathbb{P}}_s, z^{\mathbb{P}}_s)ds - \int_t^\tau z^{\mathbb{P}}_s dB_s,\ 0 \le t \le \tau,\ \mathbb{P}-a.s. \tag{3.1}$$

First, we introduce the following simple generalization of the comparison Theorem proved in [28] (see Theorem 2).

Proposition 3.1. *Let Assumptions 2.1 hold true. Let ξ_1 and $\xi_2 \in L^\infty(\mathbb{P})$ for some probability measure $\mathbb{P}$, and V^i, $i = 1,2$ be two adapted, càdlàg non-decreasing processes null at 0. Let $(Y^i, Z^i) \in \mathbb{D}^\infty(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P})$, $i = 1,2$ be the solutions of the following BSDEs*

$$Y^i_t = \xi^i + \int_t^T \widehat{F}_s(Y^i_s, Z^i_s)ds - \int_t^T Z^i_s dB_s + V^i_T - V^i_t,\ \mathbb{P}-a.s.,\ i = 1,2,$$

respectively. If $\xi_1 \ge \xi_2$, $\mathbb{P}-a.s.$ and $V^1 - V^2$ is non-decreasing, then it holds $\mathbb{P}-a.s.$ that for all $t \in [0,T]$

$$Y^1_t \ge Y^2_t.$$

Proof. First of all, we need to justify the existence of the solutions to those BSDEs. Actually, this is a simple consequence of the existence results of Tevzadze [28] and for instance Proposition 3.1 in [18]. Then, the above comparison is a mere generalization of Theorem 2 in [28]. □

We then have similarly as in Theorem 4.4 of [24] the following results which justifies the PDE intuition given in the Introduction.

Theorem 3.1. *Let Assumptions 2.2 hold. Assume $\xi \in \mathbb{L}^\infty_H$ and that $(Y,Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ is a solution to 2BSDE (2.5). Then, for any $\mathbb{P} \in \mathcal{P}_H$ and $0 \le t_1 < t_2 \le T$,*

$$Y_{t_1} = \underset{\mathbb{P}' \in \mathcal{P}_H(t_1,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ y^{\mathbb{P}'}_{t_1}(t_2, Y_{t_2}),\ \mathbb{P}-a.s. \tag{3.2}$$

Consequentlty, the 2BSDE (2.5) has at most one solution in $\mathbb{D}^\infty_H \times \mathbb{H}^2_H$.

Before proceeding with the proof, we will need the following Lemma which shows that in our 2BSDE framework, we still have a deep link between quadratic growth in z of the generator and the BMO spaces.

Lemma 3.1. *Let Assumption 2.2 hold. Assume $\xi \in \mathbb{L}^\infty_H$ and that $(Y,Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ is a solution to 2BSDE* (2.5)*. Then $Z \in \mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$.*

Proof. Denote $\mathcal{T}_0^T$ the collection of stopping times taking values in $[0,T]$. By Itô's formula under $\mathbb{P}$ applied to $e^{-\nu Y_t}$, which is a càdlàg process, for some $\nu > 0$, we have for every $\tau \in \mathcal{T}_0^T$

$$\begin{aligned}\frac{\nu^2}{2}\int_\tau^T e^{-\nu Y_t}\left|\widehat{a}_t^{1/2}Z_t\right|^2 dt &= e^{-\nu\xi} - e^{-\nu Y_\tau} - \nu\int_\tau^T e^{-\nu Y_{t^-}}dK_t^{\mathbb{P}} - \nu\int_\tau^T e^{-\nu Y_t}\widehat{F}_t(Y_t,Z_t)dt \\ &\quad + \nu\int_\tau^T e^{-\nu Y_{t^-}}Z_t dB_t - \sum_{\tau\leq s\leq T} e^{-\nu Y_s} - e^{-\nu Y_{s^-}} + \nu\Delta Y_s e^{-\nu Y_{s^-}}.\end{aligned}$$

Since $Y \in \mathbb{D}^\infty_H$, $K^{\mathbb{P}}$ is non-decreasing and since the contribution of the jumps is negative because of the convexity of the function $x \to e^{-\nu x}$, we obtain with Assumption 2.2(iv)

$$\begin{aligned}\frac{\nu^2}{2}\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T e^{-\nu Y_t}\left|\widehat{a}_t^{1/2}Z_t\right|^2 dt\right] &\leq e^{\nu\|Y\|_{\mathbb{D}^\infty_H}}\left(1+\nu T\left(\alpha+\beta\|Y\|_{\mathbb{D}^\infty_H}\right)\right) \\ &\quad + \frac{\nu\gamma}{2}\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T e^{-\nu Y_t}\left|\widehat{a}_t^{1/2}Z_t\right|^2 dt\right].\end{aligned}$$

By choosing $\nu = 2\gamma$, we then have

$$\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T e^{-2\gamma Y_t}\left|\widehat{a}_t^{1/2}Z_t\right|^2 dt\right] \leq \frac{1}{\gamma}e^{2\gamma\|Y\|_{\mathbb{D}^\infty_H}}\left(1+2\gamma T\left(\alpha+\beta\|Y\|_{\mathbb{D}^\infty_H}\right)\right).$$

Finally, we obtain

$$\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T \left|\widehat{a}_t^{1/2}Z_t\right|^2 dt\right] \leq \frac{1}{\gamma}e^{4\gamma\|Y\|_{\mathbb{D}^\infty_H}}\left(1+2\gamma T\left(\alpha+\beta\|Y\|_{\mathbb{D}^\infty_H}\right)\right),$$

which provides the result by arbitrariness of $\mathbb{P}$ and τ. □

Proof of Theorem 3.1. The proof follows the lines of the proof of Theorem 4.4 in [24], but we have to deal with some specific difficulties due to our quadratic growth assumption. First if (3.2) holds, this implies that

$$Y_t = \underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\; y_t^{\mathbb{P}'}(T,\xi),\ t\in[0,T],\ \mathbb{P}-a.s. \text{ for all } \mathbb{P}\in\mathcal{P}_H,$$

and thus is unique. Then, since we have that $d\langle Y,B\rangle_t = Z_t d\langle B\rangle_t$, $\mathcal{P}_H - q.s.$, Z is also unique. We now prove (3.2) in three steps. Roughly speaking, we will obtain one inequality using the comparison theorem, and the second one by using the minimal condition (2.7).

(i) Fix $0 \leq t_1 < t_2 \leq T$ and $\mathbb{P} \in \mathcal{P}_H$. For any $\mathbb{P}^{'} \in \mathcal{P}_H(t_1^+, \mathbb{P})$, we have

$$Y_t = Y_{t_2} + \int_t^{t_2} \widehat{F}_s(Y_s, Z_s)ds - \int_t^{t_2} Z_s dB_s + K_{t_2}^{\mathbb{P}^{'}} - K_t^{\mathbb{P}^{'}},\ t_1 \leq t \leq t_2,\ \mathbb{P}^{'} - a.s.$$

and that $K^{\mathbb{P}^{'}}$ is nondecreasing, $\mathbb{P}^{'} - a.s.$ Then, we can apply the comparison Theorem 3.1 under $\mathbb{P}^{'}$ to obtain $Y_{t_1} \geq y_{t_1}^{\mathbb{P}^{'}}(t_2, Y_{t_2})$, $\mathbb{P}^{'} - a.s.$ Since $\mathbb{P}^{'} = \mathbb{P}$ on $\mathcal{F}_t^+$, we get $Y_{t_1} \geq y_{t_1}^{\mathbb{P}^{'}}(t_2, Y_{t_2})$, $\mathbb{P} - a.s.$ and thus

$$Y_{t_1} \geq \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}^{'} \in \mathcal{P}_H(t_1^+, \mathbb{P})} y_{t_1}^{\mathbb{P}^{'}}(t_2, Y_{t_2}),\ \mathbb{P} - a.s.$$

(ii) We now prove the reverse inequality. Fix $\mathbb{P} \in \mathcal{P}_H$. Let us assume for the time being that

$$C_{t_1}^{\mathbb{P},p} := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}^{'} \in \mathcal{P}_H(t_1^+, \mathbb{P})} \mathbb{E}_{t_1}^{\mathbb{P}^{'}} \left[\left(K_{t_2}^{\mathbb{P}^{'}} - K_{t_1}^{\mathbb{P}^{'}} \right)^p \right] < +\infty,\ \mathbb{P} - a.s.,\ \text{for all } p \geq 1.$$

For every $\mathbb{P}^{'} \in \mathcal{P}_H(t^+, \mathbb{P})$, denote

$$\delta Y := Y - y^{\mathbb{P}^{'}}(t_2, Y_{t_2}) \text{ and } \delta Z := Z - z^{\mathbb{P}^{'}}(t_2, Y_{t_2}).$$

By the Lipschitz Assumption 2.2(vi) and the local Lipschitz Assumption 2.2(v), there exist a bounded process λ and a process η with

$$|\eta_t| \leq \mu \left(\left| \widehat{a}_t^{1/2} Z_t \right| + \left| \widehat{a}_t^{1/2} z_t^{\mathbb{P}^{'}} \right| \right),\ \mathbb{P}^{'} - a.s.$$

such that

$$\delta Y_t = \int_t^{t_2} \left(\lambda_s \delta Y_s + (\eta_s + \phi_s) \widehat{a}_s^{1/2} \delta Z_s \right) ds - \int_t^{t_2} \delta Z_s dB_s + K_{t_2}^{\mathbb{P}^{'}} - K_t^{\mathbb{P}^{'}},\ t \leq t_2,\ \mathbb{P}^{'} - a.s.$$

Define for $t_1 \leq t \leq t_2$

$$M_t := \exp\left(\int_{t_1}^t \lambda_s ds \right),\ \mathbb{P}^{'} - a.s.$$

Now, since $\phi \in \mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$, by Lemma 2.1, we know that the exponential martingale

$$\mathcal{E}\left(\int_0^{\cdot} (\phi_s + \eta_s) \widehat{a}_s^{-1/2} dB_s \right),$$

is a $\mathbb{P}^{'}$-uniformly integrable martingale. Therefore we can define a probability measure $\mathbb{Q}^{'}$, which is equivalent to $\mathbb{P}^{'}$, by its Radon-Nykodym derivative

$$\frac{d\mathbb{Q}^{'}}{d\mathbb{P}^{'}} = \mathcal{E}\left(\int_0^T (\phi_s + \eta_s)) \widehat{a}_s^{-1/2} dB_s \right).$$

Then, by Itô's formula, we obtain, as in [24], that

$$\delta Y_{t_1} = \mathbb{E}_{t_1}^{\mathbb{Q}^{'}} \left[\int_{t_1}^{t_2} M_t dK_t^{\mathbb{P}^{'}} \right] \leq \mathbb{E}_{t_1}^{\mathbb{Q}^{'}} \left[\sup_{t_1 \leq t \leq t_2} (M_t)(K_{t_2}^{\mathbb{P}^{'}} - K_{t_1}^{\mathbb{P}^{'}}) \right],$$

since $K^{\mathbb{P}^{'}}$ is non-decreasing.

Then, since λ is bounded, we have that M is also bounded and thus for every $p \geq 1$

$$\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\sup_{t_1 \leq t \leq t_2} (M_t)^p\right] \leq C_p, \ \mathbb{P}' - a.s. \tag{3.3}$$

This is now that the we will need more work than in the Lipschitz case, and that the BMO properties of the solution will play a crucial role. Since $(\eta + \phi)\widehat{a}_s^{-1/2}$ is in $\mathbb{BMO}(\mathcal{P}_{\mathrm{H}})$, we know by Lemma 2.1 that there exists $r > 1$, independent of the probability measure considered, such that

$$\sup_{\mathbb{P} \in \mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\left(\mathcal{E}\left(\int_0^T (\phi_s + \eta_s)\widehat{a}_s^{-1/2} dB_s\right)\right)^r\right] < +\infty.$$

Then it follows from the Hölder inequality and Bayes Theorem that

$$\begin{aligned}\delta Y_{t_1} &\leq \frac{\left(\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\mathcal{E}\left(\int_0^T (\phi_s + \eta_s)\widehat{a}_s^{-1/2} dB_s\right)^r\right]\right)^{\frac{1}{r}}}{\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\mathcal{E}\left(\int_0^T (\phi_s + \eta_s)\widehat{a}_s^{-1/2} dB_s\right)\right]} \left(\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\left(\sup_{t_1 \leq t \leq t_2} M_t\right)^q \left(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\right)^q\right]\right)^{\frac{1}{q}} \\ &\leq C \left(C_{t_1}^{\mathbb{P}, 4q-1}\right)^{\frac{1}{4q}} \left(\mathbb{E}_{t_1}^{\mathbb{P}'}\left[K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\right]\right)^{\frac{1}{4q}}.\end{aligned}$$

By the minimum condition (2.7) and since $\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})$ is arbitrary, this ends the proof.

(iii) It remains to show that the estimate for $C_{t_1}^{\mathbb{P},p}$ holds for $p \geq 1$. Once again, this will be possible because of the BMO property satisfied by the solution of the 2BSDE. By definition of the family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H\}$, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\right)^p\right] &\leq C\left(1 + \|Y\|_{\mathbb{D}_H^\infty}^p + \|\xi\|_{\mathbb{L}_H^\infty}^p + \mathbb{E}_{t_1}^{\mathbb{P}'}\left[\left(\int_{t_1}^{t_2} \left|\widehat{a}_t^{1/2} Z_t\right|^2 dt\right)^p\right]\right) \\ &\quad + C\mathbb{E}_{t_1}^{\mathbb{P}'}\left[\left(\int_{t_1}^{t_2} Z_t dB_t\right)^p\right].\end{aligned}$$

Thus by the energy inequalities (2.4) and by Burkholder-Davis-Gundy inequality, we obtain that

$$\mathbb{E}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\right)^p\right] \leq C\left(1 + \|Y\|_{\mathbb{D}_H^\infty}^p + \|\xi\|_{\mathbb{L}_H^\infty}^p + \|Z\|_{\mathbb{BMO}_{\mathrm{H}}}^{2p} + \|Z\|_{\mathbb{BMO}_{\mathrm{H}}}^p\right).$$

Therefore, we have proved that

$$\sup_{\mathbb{P}' \in \mathcal{P}_H(t_1^+, \mathbb{P})} \mathbb{E}^{\mathbb{P}'}\left[\left(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\right)^p\right] < +\infty.$$

Then we can proceed exactly as in the proof of Theorem 4.4 in [24]. □

Remark 3.1. *It is interesting to notice that in contrast with standard quadratic BSDEs, for which the only property of BMO martingales used to obtain uniqueness is the fact that their Doléans-Dade exponential is a uniformly integrable martingale, we need a lot more in the 2BSDE framework. Indeed, we use extensively the energy inequalities and the existence of*

moments for the Doléans-Dade exponential (which is a consequence of the so called reverse Hölder inequalities, which is a more general version of Lemma 2.1). Furthermore, we will also use the so-called Muckenhoupt condition (which corresponds to Lemma 2.2, see [16] for more details) in both our proofs of existence. This seems to be directly linked to the presence of the non-decreasing processes $K^{\mathbb{P}}$ and raises the question about whether it could be possible to generalize the recent approach of Barrieu and El Karoui [2], to second-order BSDEs. Indeed, since they no longer assume a bounded terminal condition, the Z part of the solution is no-longer BMO. We leave this interesting but difficult question to future research.

We conclude this section by showing some a priori estimates which will be useful in the sequel. Notice that these estimates also imply uniqueness, but they use intensively the representation formula (3.2).

Theorem 3.2. *Let Assumption 2.2 hold.*

(i) *Assume that $\xi \in \mathbb{L}^\infty_H$ and that $(Y,Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ is a solution to 2BSDE (2.5). Then, there exists a constant C such that*

$$\|Y\|_{\mathbb{D}^\infty_H} + \|Z\|^2_{\mathbb{BMO}(\mathcal{P}_\mathrm{H})} \leq C\left(1+\|\xi\|_{\mathbb{L}^\infty_H}\right)$$
$$\forall p \geq 1, \quad \sup_{\mathbb{P}\in\mathcal{P}_H,\ \tau\in\mathcal{T}_0^T} \mathbb{E}^{\mathbb{P}}_\tau\left[(K^{\mathbb{P}}_T - K^{\mathbb{P}}_\tau)^p\right] \leq C\left(1+\|\xi\|^p_{\mathbb{L}^\infty_H}\right).$$

(ii) *Assume that $\xi^i \in \mathbb{L}^\infty_H$ and that $(Y^i,Z^i) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$ is a corresponding solution to 2BSDE (2.5), $i=1,2$. Denote $\delta\xi := \xi^1-\xi^2$, $\delta Y := Y^1 - Y^2$, $\delta Z := Z^1 - Z^2$ and $\delta K^{\mathbb{P}} := K^{\mathbb{P},1} - K^{\mathbb{P},2}$. Then, there exists a constant C such that*

$$\|\delta Y\|_{\mathbb{D}^\infty_H} \leq C\,\|\delta\xi\|_{\mathbb{L}^\infty_H}$$
$$\|\delta Z\|^2_{\mathbb{BMO}(\mathcal{P}_\mathrm{H})} \leq C\,\|\delta\xi\|_{\mathbb{L}^\infty_H}\left(1+\left\|\xi^1\right\|_{\mathbb{L}^\infty_H}+\left\|\xi^2\right\|_{\mathbb{L}^\infty_H}\right)$$
$$\forall p \geq 1, \ \sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|\delta K^{\mathbb{P}}_t\right|^p\right] \leq C\,\|\xi\|^{\frac{p}{2}}_{\mathbb{L}^\infty_H}\left(1+\left\|\xi^1\right\|^{\frac{p}{2}}_{\mathbb{L}^\infty_H}+\left\|\xi^2\right\|^{\frac{p}{2}}_{\mathbb{L}^\infty_H}\right).$$

Proof. (i) By Theorem 3.1 we know that for all $\mathbb{P}\in\mathcal{P}_H$ and for all $t\in[0,T]$ we have

$$Y_t = \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})} y^{\mathbb{P}'}_t.$$

Then by Lemma 1 in [6], we know that for all $\mathbb{P}\in\mathcal{P}_H$

$$\left|y^{\mathbb{P}}_t\right| \leq \frac{1}{\gamma}\log\left(\mathbb{E}^{\mathbb{P}}_t\left[\psi(|\xi|)\right]\right), \text{ where } \psi(x) := \exp\left(\gamma\alpha\frac{e^{\beta T}-1}{\beta}+\gamma e^{\beta T}x\right).$$

Thus, we obtain $|y^{\mathbb{P}}_t| \leq \alpha\left(e^{\beta T}-1\right)/\beta + e^{\beta T}\|\xi\|_{\mathbb{L}^\infty_H}$, and by the representation recalled above, the estimate of $\|Y\|_{\mathbb{D}^\infty_H}$ is obvious. By the proof of Lemma 3.1, we have now

$$\|Z\|^2_{\mathbb{BMO}(\mathcal{P}_\mathrm{H})} \leq Ce^{C\|Y\|_{\mathbb{D}^\infty_H}}\left(1+\|Y\|_{\mathbb{D}^\infty_H}\right) \leq C\left(1+\|\xi\|_{\mathbb{L}^\infty_H}\right).$$

Finally, we have for all $\tau \in \mathcal{T}_0^T$, for all $\mathbb{P} \in \mathcal{P}_H$ and for all $p \geq 1$, by definition

$$(K_T^{\mathbb{P}} - K_\tau^{\mathbb{P}})^p = \left(Y_\tau - \xi - \int_\tau^T \widehat{F}_t(Y_y, Z_t)dt + \int_\tau^T Z_t dB_t\right)^p.$$

Therefore, by our growth Assumption 2.2(iv)

$$\begin{aligned}\mathbb{E}_\tau^{\mathbb{P}}\left[(K_T^{\mathbb{P}} - K_\tau^{\mathbb{P}})^p\right] &\leq C\left(1 + \|\xi\|_{\mathbb{L}_H^\infty}^p + \|Y\|_{\mathbb{D}_H^\infty}^p + \mathbb{E}_\tau^{\mathbb{P}}\left[\left(\int_\tau^T |\widehat{a}_t^{\frac{1}{2}} Z_t|^2 dt\right)^p + \left(\int_\tau^T Z_t dB_t\right)^p\right]\right)\\ &\leq C\left(1 + \|\xi\|_{\mathbb{L}_H^\infty}^p + \|Z\|_{\mathbb{BMO}(\mathcal{P}_H)}^{2p} + \|Z\|_{\mathbb{BMO}(\mathcal{P}_H)}^p\right) \leq C\left(1 + \|\xi\|_{\mathbb{L}_H^\infty}^p\right),\end{aligned}$$

where we used again the energy inequalities and the BDG inequality. This provides the estimate for $K^{\mathbb{P}}$ by arbitrariness of τ and $\mathbb{P}$.

(ii) With the same notations and calculations as in step (ii) of the proof of Theorem 3.1, it is easy to see that for all $\mathbb{P} \in \mathcal{P}_H$ and for all $t \in [0, T]$, we have

$$\delta y_t^{\mathbb{P}} = \mathbb{E}_t^{\mathbb{Q}}\left[M_T \delta\xi\right] \leq C \|\delta\xi\|_{\mathbb{L}_H^\infty},$$

since M is bounded and we have (3.3). By Theorem 3.1, the estimate for δY follows. Now apply Itô's formula under a fixed $\mathbb{P} \in \mathcal{P}_H$ to $|\delta Y|^2$ between $\tau \in \mathcal{T}_0^T$ and T

$$\begin{aligned}\mathbb{E}_\tau^{\mathbb{P}}\left[|\delta Y_\tau|^2 + \int_\tau^T \left|\widehat{a}_t^{1/2}\delta Z_t\right|^2 dt\right] &\leq \mathbb{E}_\tau^{\mathbb{P}}\left[|\delta\xi|^2 + 2\int_\tau^T \delta Y_t\left(\widehat{F}_t(Y_t^1, Z_t^1) - \widehat{F}_t(Y_t^2, Z_t^2)\right)dt\right]\\ &\quad - 2\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T \delta Y_{t^-} d(\delta K_t^{\mathbb{P}})\right].\end{aligned}$$

Then, we have by Assumption 2.2(iv) and the estimates proved in (i) above

$$\begin{aligned}\mathbb{E}_\tau^{\mathbb{P}}\left[\int_\tau^T \left|\widehat{a}_t^{1/2}\delta Z_t\right|^2 dt\right] &\leq C\|\delta Y\|_{\mathbb{D}_H^\infty}\left(1 + \sum_{i=1}^2 \left\|Y^i\right\|_{\mathbb{D}_H^\infty} + \left\|Z^i\right\|_{\mathbb{BMO}(\mathcal{P}_H)}\right)\\ &\quad + \|\delta\xi\|_{\mathbb{L}_H^\infty}^2 + 2\|\delta Y\|_{\mathbb{D}_H^\infty}\mathbb{E}_\tau^{\mathbb{P}}\left[\left|K_T^{\mathbb{P},1} - K_\tau^{\mathbb{P},1}\right| + \left|K_T^{\mathbb{P},2} - K_\tau^{\mathbb{P},2}\right|\right]\\ &\leq C\|\delta\xi\|_{\mathbb{L}_H^\infty}\left(1 + \left\|\xi^1\right\|_{\mathbb{L}_H^\infty} + \left\|\xi^2\right\|_{\mathbb{L}_H^\infty}\right),\end{aligned}$$

which implies the required estimate for δZ. Finally, by definition, we have for all $\mathbb{P} \in \mathcal{P}_H$ and for all $t \in [0, T]$

$$\delta K_t^{\mathbb{P}} = \delta Y_0 - \delta Y_t - \int_0^t \widehat{F}_s(Y_s^1, Z_s^1) - \widehat{F}_s(Y_s^2, Z_s^2)ds + \int_0^t \delta Z_s dB_s.$$

By Assumptions 2.2(v) and (vi), it follows that

$$\sup_{0\leq t\leq T}\left|\delta K_t^{\mathbb{P}}\right| \leq C\left(\|\delta Y\|_{\mathbb{D}_H^\infty} + \int_0^T |\widehat{a}_s^{\frac{1}{2}}\delta Z_s|(1 + |\widehat{a}_s^{\frac{1}{2}} Z_s^1| + |\widehat{a}_s^{\frac{1}{2}} Z_s^2|)ds + \sup_{0\leq t\leq T}\left|\int_0^t \delta Z_s dB_s\right|\right),$$

and by Cauchy-Schwarz, BDG and energy inequalities, we see that

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq T}\left|\delta K_t^{\mathbb{P}}\right|^p\right]\leq C\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T 1+|\widehat{a}_s^{\frac{1}{2}}Z_s^1|^2+|\widehat{a}_s^{\frac{1}{2}}Z_s^2|^2ds\right)^p\right]^{\frac{1}{2}}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T |\widehat{a}_s^{\frac{1}{2}}\delta Z_s|^2ds\right)^p\right]^{\frac{1}{2}}$$
$$+C\left(\|\delta\xi\|^p_{\mathbb{L}^\infty_H}+\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T\left|\widehat{a}_s^{1/2}\delta Z_s\right|^2ds\right)^{p/2}\right]\right)$$
$$\leq C\,\|\delta\xi\|^{p/2}_{\mathbb{L}^\infty_H}\left(1+\left\|\xi^1\right\|^{p/2}_{\mathbb{L}^\infty_H}+\left\|\xi^2\right\|^{p/2}_{\mathbb{L}^\infty_H}\right).$$

□

Remark 3.2. *Let us note that the proof of* (i) *only requires that Assumption 2.2*(iv) *holds true, whereas* (ii) *also requires Assumption 2.2*(v) *and* (vi)*.*

4 2BSDEs and monotone approximations

This Section is devoted to the study of monotone approximations in the 2BSDE framework. We start with the simplest quadratic 2BSDEs, which allows us to introduce a quasi-sure version of the entropic risk measure. In that case, we obtain existence through the classical exponential change. Then, we show that for more general generators, this approach usually fails because of the absence of a general quasi-sure monotone convergence Theorem. Finally, we prove an existence result using another type of approximation which has the property to be stationary.

4.1 Entropy and purely quadratic 2BSDEs

Given $\xi\in\mathcal{L}^\infty_H$, we first consider the purely quadratic 2BSDE defined as follows

$$Y_t=-\xi+\int_t^T\frac{\gamma}{2}\left|\widehat{a}_s^{1/2}Z_s\right|^2ds-\int_t^T Z_sdB_s+K_T^{\mathbb{P}}-K_t^{\mathbb{P}},\ 0\leq t\leq T,\ \mathcal{P}_H-q.s. \tag{4.1}$$

Then we use the classical exponential change of variables and define

$$\overline{Y}_t:=e^{\gamma Y_t},\ \overline{Z}_t:=\gamma\overline{Y}_tZ_t,\ \overline{K}_t^{\mathbb{P}}:=\gamma\int_0^t\overline{Y}_sdK_s^{\mathbb{P}}-\sum_{0\leq s\leq t}e^{\gamma Y_s}-e^{\gamma Y_{s^-}}-\gamma\Delta Y_se^{\gamma Y_{s^-}}.$$

At least formally, we see that $(\overline{Y},\overline{Z},\overline{K}^{\mathbb{P}})$ verifies the following equation

$$\overline{Y}_t=e^{-\gamma\xi}-\int_t^T\overline{Z}_sdB_s+\overline{K}_T^{\mathbb{P}}-\overline{K}_t^{\mathbb{P}},\ 0\leq t\leq T,\ \mathbb{P}-a.s.\ \forall\mathbb{P}\in\mathcal{P}_H \tag{4.2}$$

which is in fact a 2BSDE with generator equal to 0 (and thus Lipschitz), provided that the familly $\left(\overline{K}^{\mathbb{P}}\right)_{\mathbb{P}\in\mathcal{P}_H}$ satisfies the minimum condition (2.7). Thus the purely quadratic 2BSDE (4.1) is linked to the 2BSDE with Lipschitz generator (4.2), which has a unique solution by Soner, Touzi and Zhang [24]. We now make this rigorous.

Proposition 4.1. *The 2BSDE* (4.1) *has a unique solution* $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ *given by*

$$Y_t = \frac{1}{\gamma} \ln \left(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} \left[e^{-\gamma\xi} \right] \right), \ \mathbb{P} - a.s., \ t \in [0,T], \ \textit{for all } \mathbb{P} \in \mathcal{P}_H.$$

Proof. Uniqueness is a simple consequence of Theorem 3.1. In the following, we prove the existence in 3 steps.

Step 1: Let $(\overline{Y}, \overline{Z}) \in \mathbb{D}_H^2 \times \mathbb{H}_H^2$ be the unique solutinon to the 2BSDE (4.2) and $\overline{K}^{\mathbb{P}}$ be the correponding non-decreasing processes. In particular, we know that

$$\overline{Y}_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} \left[e^{-\gamma\xi} \right], \ \mathbb{P} - a.s.,$$

which implies that $\overline{Y} \in \mathbb{D}_H^\infty$, since

$$0 < e^{-\gamma \|\xi\|_{\mathbb{L}_H^\infty}} \leq Y_t \leq e^{\gamma \|\xi\|_{\mathbb{L}_H^\infty}}.$$

We can therefore make the following change of variables

$$Y_t := \frac{1}{\gamma} \ln \left(\overline{Y}_t \right), \ Z_t := \frac{1}{\gamma} \frac{\overline{Z}_t}{\overline{Y}_t}.$$

Then by Itô's formula, we can verify that the pair $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ satisfies (4.1) with

$$K_t^{\mathbb{P}} := \int_0^t \frac{1}{\gamma \overline{Y}_s} d\overline{K}_s^{\mathbb{P},c} - \sum_{0 < s \leq t} \frac{1}{\gamma} \log \left(1 - \frac{\Delta \overline{K}_s^{\mathbb{P},d}}{\overline{Y}_{s-}} \right).$$

Moreover, notice that $K^{\mathbb{P}}$ is non-decreasing with $K_0^{\mathbb{P}} = 0$.

Step 2: Denote now $(y^{\mathbb{P}}, z^{\mathbb{P}})$ the solutions of the standard BSDEs corresponding to the 2BSDE (4.1) (existence and uniqueness are ensured for example by [28]). Furthermore, if we define

$$\overline{y}_t^{\mathbb{P}} := e^{\gamma y_t^{\mathbb{P}}}, \ \overline{z}_t^{\mathbb{P}} := \gamma \overline{y}_t^{\mathbb{P}} z_t^{\mathbb{P}},$$

then we know that $(\overline{y}^{\mathbb{P}}, \overline{z}^{\mathbb{P}})$ solve the standard BSDE under $\mathbb{P}$ corresponding to (4.2). Due to the monotonicity of the function $x \to \ln(x)$ and the representation for $\overline{Y}$

$$\overline{Y}_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})} \overline{y}_t^{\mathbb{P}} = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} \left[e^{-\gamma\xi} \right], \ \mathbb{P} - a.s.,$$

we have the following representation for Y

$$Y_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})} y_t^{\mathbb{P}} = \frac{1}{\gamma} \ln \left(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} \left[e^{-\gamma\xi} \right] \right), \ \mathbb{P} - a.s.$$

Step 3: Finally, it remains to check the minimum condition for the family of non-decreasing processes $\left\{ K^{\mathbb{P}} \right\}$. Since the purely quadratic generator satisfies the Assumption 2.1, we can derive the minimum condition from the above representation for Y exactly as in the proof of Theorem 4.1 in Subsection 4.3. □

Thanks to the above result, we can define a quasi-sure (or robust) version of the entropic risk measure under volatility uncertainty

$$e_{\gamma,t}(\xi) := \frac{1}{\gamma}\ln\left(\underset{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[e^{-\gamma\xi}\right]\right),$$

where the parameter γ stands for the risk tolerance. We emphasize that, as proved in [25] (see Proposition 4.11), the solution of (4.1) is actually $\mathbb{F}$-measurable, so we also have

$$e_{\gamma,t}(\xi) := \frac{1}{\gamma}\ln\left(\underset{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}_t^{\mathbb{P}'}\left[e^{-\gamma\xi}\right]\right),$$

which in particular implies that

$$e_{\gamma,0}(\xi) = \frac{1}{\gamma}\ln\left(\sup_{\mathbb{P}\in\mathcal{P}_H}\mathbb{E}^{\mathbb{P}}\left[e^{-\gamma\xi}\right]\right).$$

More generally, by the same exponential change and arguments above, we can also prove that there exists a unique solution to 2BSDEs with terminal condition $\xi\in\mathcal{L}_H^\infty$ and the following type of quadratic growth generators $\widehat{a}^{1/2}zg(t,\omega)+h(t,\omega)+\frac{\theta}{2}\left|\widehat{a}_t^{1/2}z\right|^2$ where g and h are assumed to be bounded, adapted and uniformly continuous in ω for the $\|\cdot\|_\infty$.

4.2 Why the exponential transformation may fail in general ?

Coming back to Kobylanski [17], we know that the exponential transformation used in the previous subsection is an important tool in the study of quadratic BSDEs. However, unlike with a purely quadratic generator, in the general case the exponential change does not lead immediately to a Lipschitz BSDE. For the sake of clarity, let us consider the 2BSDE (2.5) and let us denote

$$\eta := e^{\gamma\xi},\ \overline{Y}_t := e^{\gamma Y_t},\ \overline{Z}_t := \gamma\overline{Y}_tZ_t,\ \overline{K}_t^{\mathbb{P}} := \gamma\int_0^t\overline{Y}_sdK_s^{\mathbb{P}} - \sum_{0\leq s\leq t}e^{\gamma Y_s}-e^{\gamma Y_{s^-}}-\gamma\Delta Y_se^{\gamma Y_{s^-}}.$$

Then we expect that, at least formally, if (Y,Z) is a solution of (2.5), then $(\overline{Y},\overline{Z})$ is a solution of the following 2BSDE

$$\overline{Y}_t = \eta+\gamma\int_t^T\overline{Y}_s\left(\widehat{F}_s\left(\frac{\log\overline{Y}_s}{\gamma},\frac{\overline{Z}_s}{\gamma\overline{Y}_s}\right)-\frac{\left|\widehat{a}_s^{1/2}\overline{Z}_s\right|^2}{2\gamma\overline{Y}_s^2}\right)ds-\int_t^T\overline{Z}_sdB_s+\overline{K}_T^{\mathbb{P}}-\overline{K}_t^{\mathbb{P}}. \quad (4.3)$$

Let us now define for $(t,y,z)\in[0,T]\times\mathbb{R}_+^*\times\mathbb{R}^d$,

$$G_t(\omega,y,z) := \gamma y\left(\widehat{F}_t\left(\omega,\frac{\log y}{\gamma},\frac{z}{\gamma y}\right)-\frac{\left|\widehat{a}_t^{1/2}z\right|^2}{2\gamma y^2}\right).$$

Then, despite the fact that the generator G is not Lipschitz, it is possible, as shown by Kobylanski [17], to find a sequence $(G^n)_{n\geq 0}$ of Lipschitz functions which decreases to G.

Then, it is possible, thanks to the result of [24] to define for each n the solution (Y^n, Z^n) of the corresponding 2BSDE. The idea is then to prove existence and uniqueness of a solution for the 2BSDE with generator G (and thus also for the 2BSDE (2.5)) by passing to the limit in some sense in the sequence (Y^n, Z^n).

If we then follow the usual approach for standard BSDEs, the first step is to argue that thanks to the comparison theorem (which still holds true for Lipschitz 2BSDEs, see [24]), the sequence Y^n is decreasing, and thanks to a priori estimates that it must converge $\mathcal{P}_H - q.s.$ to some process Y. And this is exactly now that the situation becomes much more complicated with 2BSDEs. Indeed, if we were in the classical framework, this convergence of Y^n together with the a priori estimates would be sufficient to prove the convergence in the usual $\mathbb{H}^2$ space, thanks to the dominated convergence theorem. However, in our case, since the norms involve the supremum over a family of probability measures, this theorem can fail (we refer the reader to Section 2.6 in [23] for more details). Therefore, we cannot obtain directly that

$$\sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{E}^{\mathbb{P}}\left[\int_0^T |Y_t^n - Y_t|^2 \, dt\right] \underset{n\to+\infty}{\longrightarrow} 0,$$

which is a crucial step in the approximation proof.

This is precisely the major difficulty when considering the 2BSDE framework. The only monotone convergence Theorem in a similar setting has been proved by Denis, Hu and Peng (see [11]). However, one need to consider random variables X^n which are regular in ω, more precisely quasi-continuous, that is to say that for every $\varepsilon > 0$, there exists an open set $\mathcal{O}^\varepsilon$ such that the X^n are continuous in ω outside $\mathcal{O}^\varepsilon$ and such that

$$\sup_{\mathbb{P}\in\mathcal{P}_H} \mathbb{P}(\mathcal{O}^\varepsilon) \leq \varepsilon.$$

Moreover, the set of probability measures considered must be weakly compact. This induces several fundamental problems when one tries to apply directly this Theorem to $(Y^n)_{n\geq 0}$.

(i) *First, if we assume that the terminal condition ξ is in $UC_b(\Omega)$, since the generator $\widehat{F}$ (and thus G^n) are uniformly continuous in ω, we can reasonably expect to be able to prove that the Y^n will be also continuous in ω, $\mathbb{P} - a.s.$, for every $\mathbb{P} \in \mathcal{P}_H$. However, this is clearly not sufficient to obtain the quasi-continuity. Indeed, for each $\mathbb{P}$, we would have a $\mathbb{P}$-negligible set outside of which the Y^n are continuous in ω. But since the probability measures are mutually singular, this does not imply the existence of the open set of the definition of quasi-continuity.*

We moreover emphasize that it is a priori a very difficult problem to show the quasi-continuity of the solution of a 2BSDE, because by definition, it is defined $\mathbb{P} - a.s.$ for every $\mathbb{P}$, and the quasi-continuity is by essence a notion related to the theory of capacities, not of probability measures.

(ii) *Next, it has been shown that if we assume that the matrices $\underline{a}^{\mathbb{P}}$ and $\overline{a}^{\mathbb{P}}$ appearing in Definition 2.1 are uniform in $\mathbb{P}$, then the set $\mathcal{P}_H$ is only weakly relatively compact. Then, we are left with two options. First, we can restrict ourselves to a closed subset of $\mathcal{P}_H$, which will therefore be weakly compact. However, as pointed out in [25], it is not possible*

to restrict arbitrarily the probability measures considered. Indeed, since the whole approach of [24] to prove existence of Lipschitz 2BSDEs relies on stochastic control and the dynamic programming equation, we need the set of processes α in the definition of $\overline{\mathcal{P}}_S$ (that is to say our set of control processes) to be stable by concatenation and bifurcation (see for instance Remark 3.1 *in [5]) in order to recover the results of [24]. And it is not clear at all to us whether it is possible to find a closed subset of $\mathcal{P}_H$ satisfying this stability properties.*

Otherwise, we could work with the weak closure of $\mathcal{P}_H$. The problem now is that the probability measures in that closure no longer satisfy necessarily the martingale representation property and the 0-1 Blumenthal law. In that case (since the filtration $\mathbb{F}$ will only be quasi-left continuous), and as already shown by El Karoui and Huang [13], we would need to redefine a solution of a 2BSDE by adding a martingale orthogonal to the canonical process. However, defining such solutions is a complicated problem outside of the scope of this paper.

We hope to have convinced the reader that because of all the reasons listed above, it seems difficult in general to prove existence of a solution to a 2BSDE using approximation arguments. However, the situation is not hopeless. Indeed, in [23], the author uses such an approach to prove existence of a solution to a 2BSDE with a generator with linear growth satisfying some monotonicity condition. The idea is that in this case it is possible to show that the sequence of approximated generators converges uniformly in (y, z), and this allows to have a control on the difference $|Y_t^n - Y_t|$ by a quantity which is regular enough to apply the monotone convergene Theorem of [11]. Nonetheless, this relies heavily on the type of approximation used and cannot a priori be extended to more general cases.

Notwithstanding this, we will show an existence result in the next subsection using an approximation which has the particularity of being stationary, which immediately solves the convergence problems that we mentioned above. This approach is based on very recent results of Briand and Elie [8] on standard quadratic BSDEs.

4.3 A stationary approximation

For technical reasons that we will explain below, we will work throughout this subsection under a subset of $\mathcal{P}_H$, which was first introduced in [26]. Namely, we will denote by Ξ the set of processes α satisfying

$$\alpha_t(\omega) = \sum_{n=0}^{+\infty}\sum_{i=1}^{+\infty} \alpha_t^{n,i}\mathbf{1}_{E_n^i}(\omega)\mathbf{1}_{[\tau_n(\omega),\tau_{n+1}(\omega))}(t),$$

where for each i and for each n, $\alpha^{n,i}$ is a bounded deterministic mapping, τ_n is an $\mathbb{F}$-stopping time with $\tau_0 = 0$, such that $\tau_n < \tau_{n+1}$ on $\{\tau_n < +\infty\}$, $\inf\{n \geq 0,\ \tau_n = +\infty\} < +\infty$, τ_n takes countably many values in some fixed $I_0 \subset [0, T]$ which is countable and dense in $[0, T]$ and for each n, $(E_i^n)_{i\geq 1} \subset \mathcal{F}_{\tau_n}$ forms a partition of Ω.

We will then consider the set $\widetilde{\mathcal{P}}_H := \{\mathbb{P}^\alpha \in \mathcal{P}_H,\ \alpha \in \Xi\}$. As shown in [25], this set satisfies the right stability properties (already mentioned in the previous subsection) so much so that the Lipschitz theory of 2BSDEs still holds when we are working $\widetilde{\mathcal{P}}_H - q.s.$ Notice that

for the sake of simplicity, we will keep the same notations for the spaces considered under $\widetilde{\mathcal{P}}_H$ or $\mathcal{P}_H$. Let us now describe the Assumptions under which we will be working

Assumption 4.1. *Let Assumption 2.2 holds, with the addition that the process ϕ in* (v) *is bounded and that the mapping F is deterministic.*

The main result of this Section is then

Theorem 4.1. *Let Assumption 4.1 hold. Assume further that $\xi \in \mathcal{L}_H^\infty$, that it is Malliavin differentiable $\widetilde{\mathcal{P}}_H - q.s.$ and that its Malliavin derivative is in $\mathbb{D}_H^\infty$. Then the 2BSDE* (2.5) *(considered $\widetilde{\mathcal{P}}_H - q.s.$) has a unique solution $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$. Moreover, the family $\{K^{\mathbb{P}},\ \mathbb{P} \in \widetilde{\mathcal{P}}_H\}$ can be aggregated.*

Proof. Uniqueness follows from Theorem 3.1, so we concentrate on the existence part. Let us define the following sequence of generators

$$F_t^n(y,z,a) := F_t\left(y, \frac{|z| \wedge n}{|z|} z, a\right), \text{ and } \widehat{F}_t^n(y,z) := F_t^n(y,z,\widehat{a}_t).$$

Then for each n, F^n is uniformly Lipschitz in (y, z) and thanks to Assumption 4.1, we can apply the result of [24] to obtain the existence of a solution (Y^n, Z^n) to the 2BSDE

$$Y_t^n = \xi + \int_t^T \widehat{F}_s^n(Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dB_s + K_T^{\mathbb{P},n} - K_t^{\mathbb{P},n},\ \mathbb{P} - a.s., \text{ for all } \mathbb{P} \in \widetilde{\mathcal{P}}_H. \quad (4.4)$$

Moreover, we have for all $\mathbb{P} \in \widetilde{\mathcal{P}}_H$ and for all $t \in [0, T]$

$$Y_t^n = \underset{\mathbb{P}' \in \widetilde{\mathcal{P}}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ y_t^{\mathbb{P},n}, \mathbb{P} - a.s., \quad (4.5)$$

where $(y^{\mathbb{P},n}, z^{\mathbb{P},n})$ is the unique solution of the Lipschitz BSDE with generator $\widehat{F}^n$ and terminal condition ξ under $\mathbb{P}$. Now, using Lemma 2.1 in [8] and its proof (see Remark 4.1 below) under each $\mathbb{P} \in \widetilde{\mathcal{P}}_H$, we know that the sequence $y^{\mathbb{P},n}$ is actually stationary. Therefore, by (4.5), this also implies that the sequence Y^n is stationary. Hence, we immediately have that Y^n converges to some Y in $\mathbb{D}_H^\infty$. Moreover, we still have the representation

$$Y_t = \underset{\mathbb{P}' \in \widetilde{\mathcal{P}}_H(t^+,\mathbb{P})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ y_t^{\mathbb{P}}, \mathbb{P} - a.s., \quad (4.6)$$

Now, identifying the martingale parts in (4.4), we also obtain that the sequence Z^n is stationary and thus converges trivially in $\mathbb{H}_H^2$ to some Z. For n large enough, we thus have

$$\widehat{F}_t^n(Y_t^n, Z_t^n) = \widehat{F}_t^n(Y_t, Z_t).$$

Besides, we have by Assumption 4.1

$$\left|\widehat{F}_t^n(Y_t, Z_t)\right| \leq \alpha + \beta\,|Y_t| + \frac{\gamma}{2}\left|\widehat{a}^{1/2}\frac{|Z_t| \wedge n}{|Z_t|} Z_t\right|^2 \leq \alpha + \beta\,|Y_t| + \frac{\gamma}{2}\left|\widehat{a}^{1/2} Z_t\right|^2,\ \widetilde{\mathcal{P}}_H - q.s.$$

Since $(Y,Z)\in\mathbb{D}^\infty_H\times\mathbb{H}^2_H$, we can apply the dominated convergence theorem for the Lebesgue measure to obtain by continuity of F that

$$\int_0^T \widehat{F}^n_s(Y^n_s,Z^n_s)ds \underset{n\to+\infty}{\longrightarrow} \int_0^T \widehat{F}_s(Y_s,Z_s)ds,\widetilde{\mathcal{P}}_H-q.s.$$

Using this result in (4.4), this implies necessarily that for each $\mathbb{P}$, $K^{\mathbb{P},n}$ converges $\mathbb{P}-a.s.$ to a non-decreasing process $K^{\mathbb{P}}$. Now, in order to verify that we indeed have obtained the solution, we need to check if the processes $K^{\mathbb{P}}$ satisfy the minimum condition (2.7). Let $\mathbb{P}\in\mathcal{P}_H$, $t\in[0,T]$ and $\mathbb{P}\in\mathcal{P}_H(t^+,\mathbb{P})$. From the proof of Theorem 3.1, we have with the same notations

$$\begin{aligned}\delta Y_t=\mathbb{E}^{\mathbb{Q}'}_t\left[\int_t^T M_t dK^{\mathbb{P}'}_t\right] &\geq \mathbb{E}^{\mathbb{Q}'}_t\left[\inf_{t\leq s\leq T}(M_s)(K^{\mathbb{P}'}_T-K^{\mathbb{P}'}_t)\right]\\ &=\frac{\mathbb{E}^{\mathbb{P}'}_t\left[\mathcal{E}\left(\int_0^T(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)\inf_{t\leq s\leq T}(M_s)(K^{\mathbb{P}'}_T-K^{\mathbb{P}'}_t)\right]}{\mathbb{E}^{\mathbb{P}'}_t\left[\mathcal{E}\left(\int_0^T(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)\right]}\end{aligned}$$

For notational convenience, denote $\mathcal{E}_t:=\mathcal{E}\left(\int_0^t(\phi_s+\eta_s)\widehat{a}_s^{-1/2}dB_s\right)$. Let r be the number given by Lemma 2.2 applied to $\mathcal{E}$. Then we estimate

$$\begin{aligned}&\mathbb{E}_t\left[K^{\mathbb{P}'}_T-K^{\mathbb{P}'}_t\right]\\ &\leq \mathbb{E}^{\mathbb{P}'}_t\left[\frac{\mathcal{E}_T}{\mathcal{E}_t}\inf_{t\leq s\leq T}(M_s)(K^{\mathbb{P}'}_T-K^{\mathbb{P}'}_t)\right]^{\frac{1}{2r-1}}\mathbb{E}^{\mathbb{P}'}_t\left[\left(\frac{\mathcal{E}_T}{\mathcal{E}_t}\inf_{t\leq s\leq T}(M_s)^{-1}\right)^{\frac{1}{2(r-1)}}(K^{\mathbb{P}'}_T-K^{\mathbb{P}'}_t)\right]^{\frac{2(r-1)}{2r-1}}\\ &\leq (\delta Y_t)^{\frac{1}{2r-1}}\left(\mathbb{E}^{\mathbb{P}'}_t\left[\left(\frac{\mathcal{E}_T}{\mathcal{E}_t}\right)^{\frac{1}{r-1}}\right]\right)^{\frac{r-1}{2r-1}}\left(\mathbb{E}^{\mathbb{P}'}_t\left[\inf_{t\leq s\leq T}(M_s)^{-\frac{2}{r-1}}\right]\mathbb{E}^{\mathbb{P}'}_t\left[(K^{\mathbb{P}'}_T-K^{\mathbb{P}'}_t)^4\right]\right)^{\frac{r-1}{2(2r-1)}}\\ &\leq C\left(\mathbb{E}^{\mathbb{P}'}_t\left[\left(K^{\mathbb{P}'}_T\right)^4\right]\right)^{\frac{r-1}{2(2r-1)}}(\delta Y_t)^{\frac{1}{2r-1}}.\end{aligned}$$

By following the arguments of the proof of Theorem 3.1 (ii) and (iii), we then deduce the minimum condition. Finally, the fact that the processes $K^{\mathbb{P}}$ can be aggegated is a direct consequence of the general aggregation result of Theorem 5.1 in [26]. □

Remark 4.1. *We emphasize that the result of Lemma* 2.1 *in [8] can only be applied when the generator is deterministic. However, even though F is indeed deterministic, $\widehat{F}$ is not, because $\widehat{a}$ is random. Nonetheless, given the particular form for the density of the quadratic variation of the canonical process we assumed in the definition of $\widetilde{\mathcal{P}}_H$, we can apply the result of Briand and Elie between the stopping times and on each set of the partition of Ω, since then $\widehat{a}$ and thus $\widehat{F}$ is indeed deterministic.*

5 A pathwise proof of existence

We have seen in the previous Section that it is usually extremely difficult to prove existence of a solution to a 2BSDE using monotone approximation techniques. Nonetheless, we have

shown in Theorem 3.1 that if a solution exists, it will necessarily verify the representation (2.7). This gives us a natural candidate for the solution as a supremum of solutions to standard BSDEs. However, since those BSDEs are all defined on the support of mutually singular probability measures, it seems difficult to define such a supremum, because of the problems raised by the negligible sets. In order to overcome this, Soner, Touzi and Zhang proposed in [24] a pathwise construction of the solution to a 2BSDE. Let us describe briefly their strategy.

The first step is to define pathwise the solution to a standard BSDE. For simplicity, let us consider first a BSDE with a generator equal to 0. Then, we know that the solution is given by the conditional expectation of the terminal condition. In order to define this solution pathwise, we can use the so-called regular conditional probability distribution (r.p.c.d. for short) of Stroock and Varadhan [27]. In the general case, the idea is similar and consists on defining BSDEs on a shifted canonical space.

Finally, we have to prove measurability and regularity of the candidate solution thus obtained, and the decomposition (2.5) is obtained through a non-linear Doob-Meyer decomposition. Our aim in this section is to extend this approach to the quadratic case.

5.1 Notations

For the convenience of the reader, we recall below some of the notations introduced in [24].

• For $0 \leq t \leq T$, denote by $\Omega^t := \left\{\omega \in C\left([t,T],\mathbb{R}^d\right),\ w(t)=0\right\}$ the shifted canonical space, B^t the shifted canonical process, $\mathbb{P}_0^t$ the shifted Wiener measure and $\mathbb{F}^t$ the filtration generated by B^t.

• For $0 \leq s \leq t \leq T$ and $\omega \in \Omega^s$, define the shifted path $\omega^t \in \Omega^t$

$$\omega_r^t := \omega_r - \omega_t,\ \forall r \in [t,T].$$

• For $0 \leq s \leq t \leq T$ and $\omega \in \Omega^s$, $\widetilde{\omega} \in \Omega^t$ define the concatenation path $\omega \otimes_t \widetilde{\omega} \in \Omega^s$ by

$$(\omega \otimes_t \widetilde{\omega})(r) := \omega_r 1_{[s,t)}(r) + (\omega_t + \widetilde{\omega}_r)1_{[t,T]}(r),\ \forall r \in [s,T].$$

• For $0 \leq s \leq t \leq T$ and a $\mathcal{F}_T^s$-measurable random variable ξ on Ω^s, for each $\omega \in \Omega^s$, define the shifted $\mathcal{F}_T^t$-measurable random variable $\xi^{t,\omega}$ on Ω^t by

$$\xi^{t,\omega}(\widetilde{\omega}) := \xi(\omega \otimes_t \widetilde{\omega}),\ \forall \widetilde{\omega} \in \Omega^t.$$

Similarly, for an $\mathbb{F}^s$-progressively measurable process X on $[s,T]$ and $(t,\omega) \in [s,T] \times \Omega^s$, the shifted process $\left\{X_r^{t,\omega}, r \in [t,T]\right\}$ is $\mathbb{F}^t$-progressively measurable.

• For a $\mathbb{F}$-stopping time τ, the r.c.p.d. of $\mathbb{P}$ (denoted $\mathbb{P}_\tau^\omega$) is a probability measure on $\mathcal{F}_T$ such that

$$\mathbb{E}_\tau^{\mathbb{P}}[\xi](\omega) = \mathbb{E}^{\mathbb{P}_\tau^\omega}[\xi],\ \text{for } \mathbb{P}-a.e.\ \omega.$$

It also induces naturally a probability measure $\mathbb{P}^{\tau,\omega}$ (that we also call the r.c.p.d. of $\mathbb{P}$) on $\mathcal{F}_T^{\tau(\omega)}$ which in particular satisfies that for every bounded and $\mathcal{F}_T$-measurable random

variable ξ

$$\mathbb{E}^{\mathbb{P}^\omega_\tau}[\xi] = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\xi^{\tau,\omega}].$$

• We define similarly as in Section 5 the set $\overline{\mathcal{P}}_S^t$, by restricting to the shifted canonical space Ω^t, and its subset $\mathcal{P}_H^t$.

• Finally, we define our "shifted" generator

$$\widehat{F}_s^{t,\omega}(\widetilde{\omega}, y, z) := F_s(\omega \otimes_t \widetilde{\omega}, y, z, \widehat{a}_s^t(\widetilde{\omega})),\ \forall (s, \widetilde{\omega}) \in [t, T] \times \Omega^t.$$

Notice that thanks to Lemma 4.1 in [25], this generator coincides for $\mathbb{P}$-a.e. ω with the shifted generator as defined above, that is to say

$$F_s(\omega \otimes_t \widetilde{\omega}, y, z, \widehat{a}_s(\omega \otimes_t \widetilde{\omega})).$$

The advantage of the chosen "shifted" generator is that it inherits the uniform continuity in ω under the $\mathbb{L}^\infty$ norm of F.

5.2 Existence when ξ is in $\mathrm{UC_b}(\Omega)$

As mentioned at the beginning of the Section, we will need to prove some measurability and regularity on our candidate solution. For this purpose, we need to assume more regularity on the terminal condition. When ξ is in $\mathrm{UC_b}(\Omega)$, by definition there exists a modulus of continuity function ρ for ξ and F in ω. Then, for any $0 \leq t \leq s \leq T,\ (y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$ and $\omega, \omega' \in \Omega,\ \tilde{\omega} \in \Omega^t$,

$$\left|\xi^{t,\omega}(\tilde{\omega}) - \xi^{t,\omega'}(\tilde{\omega})\right| \leq \rho\left(\left\|\omega - \omega'\right\|_t\right) \text{ and } \left|\widehat{F}_s^{t,\omega}(\tilde{\omega}, y, z) - \widehat{F}_s^{t,\omega'}(\tilde{\omega}, y, z)\right| \leq \rho\left(\left\|\omega - \omega'\right\|_t\right),$$

where $\|\omega\|_t := \sup_{0 \leq s \leq t} |\omega_s|,\ 0 \leq t \leq T$.

To prove existence, as in [24], we define the following value process V_t pathwise

$$V_t(\omega) := \sup_{\mathbb{P} \in \mathcal{P}_H^t} \mathcal{Y}_t^{\mathbb{P},t,\omega}(T, \xi),\ \text{for all } (t, \omega) \in [0, T] \times \Omega, \tag{5.1}$$

where, for any $(t_1, \omega) \in [0, T] \times \Omega,\ \mathbb{P} \in \mathcal{P}_H^{t_1},\ t_2 \in [t_1, T]$, and any $\mathcal{F}_{t_2}$-measurable $\eta \in \mathbb{L}^\infty(\mathbb{P})$, we denote $\mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2, \eta) := y_{t_1}^{\mathbb{P},t_1,\omega}$, where $\left(y^{\mathbb{P},t_1,\omega}, z^{\mathbb{P},t_1,\omega}\right)$ is the solution of the following BSDE on the shifted space Ω^{t_1} under $\mathbb{P}$

$$y_s^{\mathbb{P},t_1,\omega} = \eta^{t_1,\omega} + \int_s^{t_2} \widehat{F}_r^{t_1,\omega}\left(y_r^{\mathbb{P},t_1,\omega}, z_r^{\mathbb{P},t_1,\omega}\right) dr - \int_s^{t_2} z_r^{\mathbb{P},t_1,\omega} dB_r^{t_1},\ s \in [t_1, t_2],\ \mathbb{P}-\text{a.s.} \tag{5.2}$$

We recall that since the Blumenthal zero-one law holds for all our probability measures, $\mathcal{Y}_t^{\mathbb{P},t,\omega}(1, \xi)$ is constant for any given (t, ω) and $\mathbb{P} \in \mathcal{P}_H^t$. Therefore, the process V is well defined. However, we still do not know anything about its measurability. The following Lemma answers this question and explains the uniform continuity Assumptions in ω we made.

Lemma 5.1. *Let $\xi \in \mathrm{UC_b}(\Omega)$. Under Assumption 2.1 or Assumption 2.2 with the addition that the $\mathbb{L}^\infty_H$-norms of ξ and $\widehat{F}^0$ are small enough, we have*

$$|V_t(\omega)| \leq C\left(1 + \|\xi\|_{\mathbb{L}^\infty_H}\right), \text{ for all } (t,\omega) \in [0,T] \times \Omega.$$

Furthermore, $|V_t(\omega) - V_t(\omega')| \leq C\rho\left(\|\omega - \omega'\|_t\right)$, for all $(t,\omega,\omega') \in [0,T]\times\Omega^2$. In particular, V_t is $\mathcal{F}_t$-measurable for every $t \in [0,T]$.

Proof. (i) For each $(t,\omega) \in [0,T]\times\Omega$ and $\mathbb{P} \in \mathcal{P}^t_H$, note that for $s \in [t,T]$, we have $\mathbb{P} - a.s.$

$$y^{\mathbb{P},t,\omega}_s = \xi^{t,\omega} - \int_s^T \left[\widehat{F}^{t,\omega}_r(0) + \lambda_r y^{\mathbb{P},t,\omega}_r + (\eta_r + \phi_r)\left(\widehat{a}^t_r\right)^{1/2} z^{\mathbb{P},t,\omega}_r\right] dr - \int_s^T z^{\mathbb{P},t,\omega}_r dB^t_r,$$

where λ is bounded and η satisfies

$$|\eta_r| \leq \mu \left|\widehat{a}^{1/2}_t z^{\mathbb{P},t,\omega}_r\right|, \ \mathbb{P} - a.s.$$

Then proceeding exactly as in the second step of the proof of Theorem 3.1, we can define a bounded process M and a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that

$$\left|y^{\mathbb{P},t,\omega}_t\right| \leq \mathbb{E}^{\mathbb{Q}}_t\left[M_T\left|\xi^{t,\omega}\right|\right] \leq C\left(1 + \|\xi\|_{\mathbb{L}^\infty_H}\right).$$

By arbitrariness of $\mathbb{P}$, we get $|V_t(\omega)| \leq C(1 + \|\xi\|_{\mathbb{L}^\infty_H})$.

(ii) The proof is exactly the same as above, except that we need to use uniform continuity in ω of $\xi^{t,\omega}$ and $\widehat{F}^{t,\omega}$. In fact, if we define for $(t,\omega,\omega') \in [0,T]\times\Omega^2$

$$\delta y := y^{\mathbb{P},t,\omega} - y^{\mathbb{P},t,\omega'},\ \delta z := z^{\mathbb{P},t,\omega} - z^{\mathbb{P},t,\omega'},\ \delta\xi := \xi^{t,\omega} - \xi^{t,\omega'},\ \delta\widehat{F} := \widehat{F}^{t,\omega} - \widehat{F}^{t,\omega'},$$

then we get with the same notations

$$|\delta y_t| = \mathbb{E}^{\mathbb{Q}}\left[M_T\delta\xi + \int_t^T M_s\delta\widehat{F}_s ds\right] \leq C\rho(\left\|\omega - \omega'\right\|_t).$$

We get the result by arbitrariness of $\mathbb{P}$. □

Then, we show the same dynamic programming principle as Proposition 4.7 in [25]

Proposition 5.1. *Let $\xi \in \mathrm{UC_b}(\Omega)$. Under Assumption 2.1 or Assumption 2.2 with the addition that the $\mathbb{L}^\infty_H$-norms of ξ and $\widehat{F}^0$ are small enough, we have for all $0 \leq t_1 < t_2 \leq T$ and for all $\omega \in \Omega$*

$$V_{t_1}(\omega) = \sup_{\mathbb{P}\in\mathcal{P}^{t_1}_H} \mathcal{Y}^{\mathbb{P},t_1,\omega}_{t_1}(t_2, V^{t_1,\omega}_{t_2}).$$

The proof is almost the same as the proof in [25], but we give it for the convenience of the reader.

Proof. Without loss of generality, we can assume that $t_1 = 0$ and $t_2 = t$. Thus, we have to prove

$$V_0(\omega) = \sup_{\mathbb{P}\in\mathcal{P}_H} \mathcal{Y}^{\mathbb{P}}_0(t, V_t).$$

Denote $(y^{\mathbb{P}}, z^{\mathbb{P}}) := (\mathcal{Y}^{\mathbb{P}}(T,\xi), \mathcal{Z}^{\mathbb{P}}(T,\xi))$

(i) For any $\mathbb{P} \in \mathcal{P}_H$, it follows from Lemma 4.3 in [25], that for $\mathbb{P}-a.e.\ \omega \in \Omega$, the r.c.p.d. $\mathbb{P}^{t,\omega} \in \mathcal{P}_H^t$. By Tevzadze [28], we know that when the terminal condition and $\widehat{F}^0$ are small, quadratic BSDEs whose generator satisfies Assumption (2.2) can be constructed via Picard iteration. Thus, it means that at each step of the iteration, the solution can be formulated as a conditional expectation under $\mathbb{P}$. Then, for general terminal conditions, Tevzadze showed that if the generator satisfies Assumption (2.1) (v), the solution of the quadratic BSDE can be written as a sum of quadratic BSDEs with small terminal condtion. By the properties of the r.p.c.d., this implies that

$$y_t^{\mathbb{P}}(\omega) = \mathcal{Y}_t^{\mathbb{P}^{t,\omega},t,\omega}(T,\xi), \text{ for } \mathbb{P}-a.e.\ \omega \in \Omega.$$

By definition of V_t and the comparison principle for quadratic BSDEs, we deduce that $y_0^{\mathbb{P}} \leq \mathcal{Y}_0^{\mathbb{P}}(t, V_t)$ and it follows from the arbitrariness of $\mathbb{P}$ that

$$V_0(\omega) \leq \sup_{\mathbb{P}\in\mathcal{P}_H} \mathcal{Y}_0^{\mathbb{P}}(t, V_t).$$

(ii) For the other inequality, we proceed as in [25]. Let $\mathbb{P} \in \mathcal{P}_H$ and $\epsilon > 0$. The idea is to use the definition of V as a supremum to obtain an ϵ-optimizer. However, since V depends obviously on ω, we have to find a way to control its dependence in ω by restricting it in a small ball. But, since the canonical space is separable, this is easy. Indeed, there exists a partition $(E_t^i)_{i\geq 1} \subset \mathcal{F}_t$ such that $\|\omega - \omega'\|_t \leq \epsilon$ for any i and any $\omega, \omega' \in E_t^i$.

Now for each i, fix an $\widehat{\omega}_i \in E_t^i$ and let, as advocated above, $\mathbb{P}_t^i$ be an $\epsilon-$optimizer of $V_t(\widehat{\omega}_i)$. If we define for each $n \geq 1$, $\mathbb{P}^n := \mathbb{P}^{n,\epsilon}$ by

$$\mathbb{P}^n(E) := \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^n \mathbb{E}^{\mathbb{P}_t^i}\left[1_E^{t,\omega}\right] 1_{E_t^i}\right] + \mathbb{P}(E \cap \widehat{E}_t^n), \text{ where } \widehat{E}_t^n := \cup_{i>n} E_t^i,$$

then, by the proof of Proposition 4.7 in [25], we know that $\mathbb{P}^n \in \mathcal{P}_H$ and that

$$V_t \leq y_t^{\mathbb{P}^n} + \epsilon + C\rho(\epsilon), \ \mathbb{P}^n - a.s. \text{ on } \cup_{i=1}^n E_t^i.$$

Let now $(y^n, z^n) := (y^{n,\epsilon}, z^{n,\epsilon})$ be the solution of the following BSDE on $[0,t]$

$$y_s^n = \left[y_t^{\mathbb{P}^n} + \epsilon + C\rho(\epsilon)\right] 1_{\cup_{i=1}^n E_t^i} + V_t 1_{\widehat{E}_t^n} + \int_s^t \widehat{F}_r(y_r^n, z_r^n)dr - \int_s^t z_r^n dB_r, \ \mathbb{P}^n - a.s. \quad (5.3)$$

Note that since $\mathbb{P}^n = \mathbb{P}$ on $\mathcal{F}_t$, the equality (5.3) also holds $\mathbb{P}-a.s.$ By the comparison theorem, we know that $\mathcal{Y}_0^{\mathbb{P}}(t, V_t) \leq y_0^n$. Using the same arguments and notations as in the proof of Lemma 5.1, we obtain

$$\left|y_0^n - y_0^{\mathbb{P}^n}\right| \leq C\mathbb{E}^{\mathbb{Q}}\left[\epsilon + \rho(\epsilon) + \left|V_t - y_t^{\mathbb{P}^n}\right| 1_{\widehat{E}_t^n}\right].$$

Then, by Lemma 5.1, we have

$$\mathcal{Y}_0^{\mathbb{P}}(t, V_t) \leq y_0^n \leq V_0(\omega) + C\left(\epsilon + \rho(\epsilon) + \mathbb{E}^{\mathbb{Q}}\left[\Lambda 1_{\widehat{E}_t^n}\right]\right).$$

The result follows from letting n go to $+\infty$ and ϵ to 0. $\square$

Remark 5.1. *We want to emphasize here that it is only because of this Proposition proving the dynamic programming equation that we had to consider Tevzadze [28] approach to quadratic BSDEs, instead of the more classical approach of Kobylanski [17]. Indeed, as pointed out in the proof, for technical reasons we want to be able to construct solutions of BSDEs via Picard iterations, to build upon the known properties of the r.c.p.d. Using the Assumptions 2.1 or 2.2 with the addition that the $\mathbb{L}^\infty_H$-norms of ξ and $\widehat{F}^0$ are small enough, this allows us to recover this property.*

Now that we solved the measurability issues for V_t, we need to study its regularity in time. However, it seems difficult to obtain a result directly, given the definition of V. This is the reason why we define now for all (t,ω), the $\mathbb{F}^+$-progressively measurable process

$$V_t^+ := \overline{\lim_{r\in\mathbb{Q}\cap(t,T], r\downarrow t}} V_r.$$

This new value process will then be proved to be càdlàg. Notice that a priori V^+ is only $\mathbb{F}^+$-progressively measurable, and not $\mathbb{F}$-progressively measurable. This explains why in the definition of the spaces in Section 2.3.1, the processes are assumed to be $\mathbb{F}^+$-progressively measurable.

Lemma 5.2. *Under the conditions of the previous Proposition, we have*

$$V_t^+ = \lim_{r\in\mathbb{Q}\cap(t,T], r\downarrow t} V_r,\ \mathcal{P}_H - q.s.$$

and thus V^+ is càdlàg $\mathcal{P}_H - q.s.$

Proof. Actually, we can proceed exactly as in the proof of Lemma 4.8 in [25], since the theory of g-expectations of Peng has been extended by Ma and Yao in [18] to the quadratic case (see in particular their Corollary 5.6 for our purpose). □

Finally, proceeding exactly as in Steps 1 and 2 of the proof of Theorem 4.5 in [25], and in particular using the Doob-Meyer decomposition proved in [18] (Theorem 5.2), we can get the existence of a universal process Z and a family of non-decreasing processes $\{K^{\mathbb{P}}, \mathbb{P}\in\mathcal{P}_H\}$ such that

$$V_t^+ = V_0^+ + \int_0^t \widehat{F}_s(V_s^+, Z_s)ds + \int_0^t Z_s dB_s - K_t^{\mathbb{P}},\ \mathbb{P}-a.s.\ \forall\mathbb{P}\in\mathcal{P}_H.$$

For the sake of completeness, we provide the representation (3.2) for V and V^+, and that, as shown in Proposition 4.11 of [25], we actually have $V = V^+$, $\mathcal{P}_H - q.s.$, which shows that in the case of a terminal condition in $UC_b(\Omega)$, the solution of the 2BSDE is actually $\mathbb{F}$-progressively measurable. This will be important in Section 7.

Proposition 5.2. *Let $\xi\in UC_b(\Omega)$. Under Assumption 2.1 or Assumption 2.2 with the addition that the $\mathbb{L}^\infty_H$-norms of ξ and $\widehat{F}^0$ are small enough, we have*

$$V_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t,\mathbb{P})} \mathcal{Y}_t^{\mathbb{P}'}(T,\xi) \text{ and } V_t^+ = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H(t^+,\mathbb{P})} \mathcal{Y}_t^{\mathbb{P}'}(T,\xi),\ \mathbb{P}-a.s.,\ \forall\mathbb{P}\in\mathcal{P}_H.$$

Besides, we also have for all t, $V_t = V_t^+$, $\mathcal{P}_H - q.s.$

Proof. The proof for the representations is the same as the proof of proposition 4.10 in [25], since we also have a stability result for quadratic BSDEs under our assumptions. For the equality between V and V^+, we also refer to the proof of Proposition 4.11 in [25]. $\square$

To be sure that we have found a solution to our 2BSDE, it remains to check that the family of non-decreasing processes above satisfies the minimum condition. Let $\mathbb{P} \in \mathcal{P}_H$, $t \in [0,T]$ and $\mathbb{P} \in \mathcal{P}_H(t^+, \mathbb{P})$. From the proof of Theorem 3.1, we have with the same notations

$$\delta V_t = \mathbb{E}_t^{\mathbb{Q}'}\left[\int_t^T M_t dK_t^{\mathbb{P}'}\right] \geq \mathbb{E}_t^{\mathbb{Q}'}\left[\inf_{t\leq s\leq T}(M_s)(K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'})\right]$$

$$= \frac{\mathbb{E}_t^{\mathbb{P}'}\left[\mathcal{E}\left(\int_0^T (\phi_s + \eta_s)\widehat{a}_s^{-1/2} dB_s\right) \inf_{t\leq s\leq T}(M_s)(K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'})\right]}{\mathbb{E}_t^{\mathbb{P}'}\left[\mathcal{E}\left(\int_0^T (\phi_s + \eta_s)\widehat{a}_s^{-1/2} dB_s\right)\right]}$$

We can proceed exactly as in the proof of Theorem 4.1 in Subsection 4.3. $\square$

Remark 5.2. *In order to prove the minimum condition it is fundamental that the process M above is bounded from below. For instance, it would not be the case if we had replaced the Lipschitz assumption on y by a monotonicity condition as in [23].*

5.3 Main result

We are now in position to state the main result of this section

Theorem 5.1. *Let $\xi \in \mathcal{L}_H^\infty$. Under Assumption 2.1 or Assumption 2.2 with the addition that the $\mathbb{L}_H^\infty$-norms of ξ and $\widehat{F}^0$ are small enough, there exists a unique solution $(Y, Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ of the* 2BSDE (2.5).

Proof. For $\xi \in \mathcal{L}_H^\infty$, there exists $\xi_n \in \mathrm{UC_b}(\Omega)$ such that $\|\xi - \xi_n\| \underset{n\to+\infty}{\to} 0$. Then, thanks to the a priori estimates obtained in Proposition 3.2, we can proceed exactly as in the proof of Theorem 4.6 (ii) in [24] to obtain the solution as a limit of the solution of the 2BSDE (2.5) with terminal condition ξ_n. $\square$

6 An application to robust risk-sensitive control

One application of classical quadratic BSDEs is to study risk-sensitive control problems, see El Karoui, Hamadène et Matoussi [14] for more details. In this section, we will consider a robust version of these problems.

First of all, for technical reasons, we restrict the probability measures in $\widetilde{\mathcal{P}}_H := \widetilde{\mathcal{P}}_S \bigcap \mathcal{P}_H$, where $\widetilde{\mathcal{P}}_S$ is defined in Subsection 2.1. Then $\widehat{a}$ is uniformly bounded by some $\overline{a}$, $\underline{a} \in \mathbb{S}_d^{>0}$.

For each $\mathbb{P} \in \widetilde{\mathcal{P}}_H$, we can define a $\mathbb{P}$-Brownian motion $W^{\mathbb{P}}$ by

$$dW_t^{\mathbb{P}} = \widehat{a}_t^{-1/2} dB_t \ \mathbb{P} - a.s.$$

Let us now consider some system, whose evolution is decribed (for simplicity) by the canonical process B. A controller then intervenes on the system via an adapted stochastic process

u which takes its values in a compact metric space U. The set of those controls is called admissible and denoted by $\mathcal{U}$. When the controller acts with u under the probability $\mathbb{P} \in \widetilde{\mathcal{P}}_H$, the dynamic of the controlled system remains the same, but now under the probability measure $\mathbb{P}^u$ defined by its density with respect to $\mathbb{P}$

$$\frac{d\mathbb{P}^u}{d\mathbb{P}} = \exp\left(\int_0^T \widehat{a}_t^{-1/2} g(t, B_\cdot, u_t) dW_t^{\mathbb{P}} - \frac{1}{2}\int_0^T \left|\widehat{a}_t^{-1/2} g(t, B_\cdot, u_t)\right|^2 dt\right),$$

where $g(t, \omega, u)$ is assumed to be bounded, continuous with respect to u, adapted and uniformly continuous in ω. Notice that this probability measure is well defined since $\widehat{a}$ is uniformly bounded.

Then, under $\mathbb{P}^u$, the dynamic of the system is given by

$$dB_t = g(t, B_\cdot, u_t)dt + \widehat{a}_t^{1/2} dW^{\mathbb{P},u}, \ \mathbb{P}^u - a.s.$$

where $W^{\mathbb{P},u}$ is a Brownian motion under $\mathbb{P}^u$ defined by

$$dW_t^{\mathbb{P},u} = dW_t^{\mathbb{P}} - \widehat{a}_t^{-1/2} g(t, B_\cdot, u_t)dt.$$

When the controller is risk seeking, we assume that the reward functional of the control action is given by the following expression

$$\forall u \in \mathcal{U}, \ J(u) := \sup_{\mathbb{P} \in \widetilde{\mathcal{P}}_H} \mathbb{E}^{\mathbb{P},u}\left[\exp\left(\theta \int_0^T h(s, B_\cdot, u_s)ds + \Psi(B_T)\right)\right]$$

where $\theta > 0$ is a real parameter which represents the sensitiveness of the controller with respect to risk. Here $h(t, \omega, u)$ is assumed to adapted and continuous in u, and both Ψ and h are assumed to be bounded and uniformly continuous in ω for the $\|\cdot\|_\infty$ norm. We are interested in finding an admissible control u^* which maximizes the reward $J(u)$ for the controller.

We begin with establishing the link between $J(u)$ and 2BSDEs in the following proposition

Proposition 6.1. *There exists a unique solution (Y^u, Z^u) of the 2BSDE associated with the generator $zg(t, B_\cdot, u_t) + h(t, B_\cdot, u_t) + \frac{\theta}{2}|\widehat{a}_t^{1/2} z|^2$, i.e., $\mathbb{P} - a.s.$, for all $\mathbb{P} \in \widetilde{\mathcal{P}}_H$*

$$Y_t^u = \Psi(B_T) + \int_t^T \left(Z_s^u g(s, B_\cdot, u_s) + h(s, B_\cdot, u_s) + \frac{\theta}{2}|\widehat{a}_s^{1/2} Z_s^u|^2\right) ds - \int_t^T Z_s^u dB_s - dK_t^{u,\mathbb{P}}. \tag{6.1}$$

Moreover $J(u) = \exp(\theta Y_0^u)$.

Proof. With our assumptions on g, h and Ψ, we know that the generator satisfies the Assumption 2.1, therefore there exists a unique solution to the 2BSDE (6.1). According to [14], the solution to the classical BSDE with the same terminal condition and generator as the 2BSDE (6.1) under each $\mathbb{P}$ is

$$y_t^{u,\mathbb{P}} = \frac{1}{\theta} \ln\left(\mathbb{E}_t^{\mathbb{P},u}\left[\exp\left(\theta \int_t^T h(s, B_\cdot, u_s)ds + \Psi(B_T)\right)\right]\right), \ \mathbb{P} - a.s.$$

Then by the representation for Y^u, we have

$$Y_t^u = \frac{1}{\theta} \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \widetilde{\mathcal{P}}_H(t^+,\mathbb{P})} \ln\left(\mathbb{E}_t^{\mathbb{P},u}\left[\exp\left(\theta \int_t^T h(s,B.,u_s)ds + \Psi(B_T)\right)\right]\right), \mathbb{P}-a.s.$$

Since the functional $\ln(x)$ is monotone non-decreasing, then

$$Y_t^u = \frac{1}{\theta} \ln\left(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \widetilde{\mathcal{P}}_H(t^+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'^u}\left[\exp\left(\theta \int_t^T h(s,B.,u_s)ds + \Psi(B_T)\right)\right]\right),\ \mathbb{P}-a.s.$$

Therefore, we have $J(u) = \exp\{\theta Y_0^u\}$. □

As explained in [14], by applying Benes' selection theorem, there exists a measurable version $u^*(t,B.,z)$ of

$$\arg\max I(t,B.,z,u) := zg(t,B.,u) + h(t,B.,u).$$

We know that $I^*(t,B.,z) := \sup_{u\in U} I(t,B.,z,u) = I(t,B.,z,u^*(t,B.,z))$ is convex uniformly Lipschitz in z because it is the supremum of functions which are linear in z. So the mapping $z \to I^*(t,B.,z) + \frac{1}{2}|\widehat{a}_t^{1/2} z|^2$ is continuous with quadratic growth, implying that a solution $(y^{*,\mathbb{P}}, z^{*,\mathbb{P}})$ of the BSDE associated to this generator exists. Then we have

Theorem 6.1. *There exists a unique solution (Y^*, Z^*) to the following 2BSDE*

$$Y_t^* = \Psi(B_T) + \int_t^T \left(I^*(s,B.,Z_s^*) + \frac{\theta}{2}|\widehat{a}_s^{1/2} Z_s^*|^2\right) ds - \int_t^T Z_s^* dB_s + K_T^{*,\mathbb{P}} - K_t^{*,\mathbb{P}}. \quad (6.2)$$

The admissible control $u^* := (u^*(t,B.,Z_t^*))_{t\leq T}$ is optimal and $(\exp(Y_t^*))_{t\leq T}$ is the value function of the robust risk-sensitive control problem, i.e., for any $t \leq T$ we have:

$$\exp(Y_t^*) = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t^+,\mathbb{P})} \operatorname*{ess\,sup}^{\mathbb{P}}_{u\in\mathcal{U}} \mathbb{E}_t^{\mathbb{P}',u}\left[\exp\left(\theta \int_t^T h(s,B.,u_s)ds + \Psi(B_T)\right)\right].$$

Proof. First, we need to prove the existence of a solution to the quadratic 2BSDE (6.2). Unlike in Proposition 6.1, here u^* also depends on z, so we do not know whether I^* is twice differentiable with respect to z. Therefore the generator of the 2BSDE may not satisfy the Assumption 2.1. But it's easy to see that it always satisfies the weaker Assumption 2.2, and we only need this Assumption to have uniqueness of the solution. Moreover, it was also the only one used to prove the minimum condition for the familly of non-decreasing processes in Subsection 5.2. Therefore, exactly as in Section 4, for $\mathbb{P} \in \widetilde{\mathcal{P}}_H$, by making the exponential change

$$\overline{Y}_t := e^{\theta Y_t^*},\ \overline{Z}_t := \theta \overline{Y}_t Z_t^*,\ \overline{K}_t^{\mathbb{P}} := \theta \int_0^t \overline{Y}_s dK_s^{*,\mathbb{P}} - \sum_{0\leq s\leq t} e^{\theta Y_s^*} - e^{\theta Y_{s^-}^*} - \theta \Delta Y_s^* e^{\theta Y_{s^-}^*},$$

we see that $(\overline{Y}, \overline{Z}, \overline{K}^{\mathbb{P}})$ formally verifies the following equation

$$\overline{Y}_t = e^{\theta\Psi(B_T)} + \int_t^T \sup_{u\in U}\left\{\overline{Z}_s g(s,B.,u) + \theta \overline{Y}_s h(s,B.,u)\right\} ds - \int_t^T \overline{Z}_s dB_s + \overline{K}_T^{\mathbb{P}} - \overline{K}_t^{\mathbb{P}},\ \mathbb{P}-a.s. \quad (6.3)$$

Since this is 2BSDE with Lipschitz generator from Soner, Touzi and Zhang [24], we know that $(\overline{Y}, \overline{Z}, \overline{K}^{\mathbb{P}})$ exists, is unique and satisfies the representation property (3.2). Arguing exactly as in Subsection 4.1 for the purely quadratic 2BSDEs, we can then obtain the existence. Now, from [14], we have that

$$\exp\left(y_t^{*,\mathbb{P}}\right) = \operatorname*{ess\,sup}_{u\in\mathcal{U}}{}^{\mathbb{P}} \ \mathbb{E}_t^{\mathbb{P}^u}\left[\exp\left(\theta\int_t^T h(s,B_\cdot,u_s)ds + \Psi(B_T)\right)\right].$$

Then the representation for Y^* implies the desired result. $\square$

7 Connection with fully nonlinear PDEs

In this section, we assume that all the randomness in H only depends on the current value of the canonical process B (the so-called Markov property)

$$H_t(\omega, y, z, \gamma) = h(t, B_t(\omega), y, z, \gamma),$$

where $h : [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times D_h \to \mathbb{R}$ is a deterministic map. Then, we define as in Section 5 the corresponding conjugate and bi-conjugate functions

$$f(t,x,y,z,a) := \sup_{\gamma\in D_h}\left\{\frac{1}{2}\mathrm{Tr}\left[a\gamma\right] - h(t,x,y,z,\gamma)\right\} \tag{7.1}$$

$$\widehat{h}(t,x,y,z,\gamma) := \sup_{a\in\mathbb{S}_d^{>0}}\left\{\frac{1}{2}\mathrm{Tr}\left[a\gamma\right] - f(t,x,y,z,a)\right\} \tag{7.2}$$

We denote $\mathcal{P}_h := \mathcal{P}_H$, and following [24], we strengthen Assumptions 2.1 and 2.2

Assumption 7.1. (i) *The domain D_{f_t} of f in a is independent of (x,y,z).*

(ii) *On D_{f_t}, f is uniformly continuous in t, uniformly in a.*

(iii) *f is continuous in z and there exists (α,β,γ) such that*

$$|f(t,x,y,z,a)| \le \alpha + \beta\,|y| + \frac{\gamma}{2}|a^{1/2}z|^2, \text{ for all } (t,x,y,z,a).$$

(iv) *f is uniformly continuous in x, uniformly in (t,y,z,a), with a modulus of continuity ρ which has polynomial growth.*

(v) *f is C^1 in y and C^2 in z, and there are constants r and θ such that for all (t,x,y,z,a)*

$$|D_y f(t,x,y,z,a)| \le r,\ |D_z f(t,x,y,z,a)| \le r + \theta|a^{1/2}z|,\ |D^2_{zz}f(t,x,y,z,a)| \le \theta.$$

Assumption 7.2. *Let points* (i) *through* (iv) *hold and*

(v) *There exists $\mu > 0$ and a bounded $\mathbb{R}^d$-valued function ϕ such that for all (t,y,z,z',a)*

$$|f(t,x,y,z,a) - f(t,x,y,z',a) - \phi(t).a^{1/2}(z-z')| \le \mu a^{1/2}\left|z-z'\right|\left(\left|a^{1/2}z\right| + \left|a^{1/2}z'\right|\right).$$

(vi) *f is Lipschitz in y, uniformly in (t,x,z,a).*

Let now $g : \mathbb{R}^d \to \mathbb{R}$ be a Lebesgue measurable and bounded function. Our object of interest here is the following 2BSDE with terminal condition $\xi = g(B_T)$

$$Y_t = g(B_T) + \int_t^T f(s, B_s, Y_s, Z_s, \widehat{a}_s^{1/2})ds - \int_t^T Z_s dB_s + K_T^{\mathbb{P}} - K_t^{\mathbb{P}},\ \mathcal{P}_h - q.s. \tag{7.3}$$

The aim of this section is to generalize the results of [24] and obtain the connection $Y_t = v(t, B_t)$, $\mathcal{P}_h - q.s.$, where v verifies in some sense the following fully nonlinear PDE

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) + \widehat{h}\left(t, x, v(t,x), Dv(t,x), D^2 v(t,x)\right) = 0,\ t \in [0,T) \\ v(T,x) = g(x). \end{cases} \tag{7.4}$$

Following the classical terminology in the BSDE litterature, we say that the solution of the 2BSDE is Markovian if it can be represented by a deterministic function of t and B_t. In this subsection, we will construct such a function following the same spirit as in the construction in the previous section. We want to emphasize that the proofs here follow very closely the proofs in [24], so we will sometimes only sketch them.

With the same notations for shifted spaces, we define for any $(t,x) \in [0,T] \times \mathbb{R}^d$

$$B_s^{t,x} := x + B_s^t,\ \text{for all } s \in [t,T].$$

Let now τ be an $\mathbb{F}^t$-stopping time, $\mathbb{P} \in \mathcal{P}_h^t$ and η a $\mathbb{P}$-bounded $\mathcal{F}_\tau^t$-measurable random variable. Similarly as in (5.2), we denote $(y^{\mathbb{P},t,x}, z^{\mathbb{P},t,x}) := (\mathcal{Y}^{\mathbb{P},t,x}(\tau,\eta), \mathcal{Z}^{\mathbb{P},t,x}(\tau,\eta))$ the unique solution of the following BSDE

$$y_s^{\mathbb{P},t,x} = \eta + \int_s^\tau f(u, B_u^{t,x}, y_u^{\mathbb{P},t,x}, z_u^{\mathbb{P},t,x}, \widehat{a}_u^t)du - \int_s^\tau z_u^{\mathbb{P},t,x} dB_u^{t,x},\ t \le s \le \tau,\ \mathbb{P} - a.s. \tag{7.5}$$

Next, we define the following deterministic function (by virtue of the Blumenthal $0-1$ law)

$$u(t,x) := \sup_{\mathbb{P} \in \mathcal{P}_h^t} \mathcal{Y}_t^{\mathbb{P},t,x}(T, g(B_T^{t,x})),\ \text{for } (t,x) \in [0,T] \times \mathbb{R}^d. \tag{7.6}$$

We then have the following Theorem, which is actually Theorem 5.9 of [24] in our framework

Theorem 7.1. *Let Assumption 7.1 hold, and assume that g is bounded and uniformly continuous. Then the 2BSDE (7.3) has a unique solution $(Y,Z) \in \mathbb{D}_H^\infty \times \mathbb{H}_H^2$ and we have $Y_t = u(t, B_t)$. Moreover, u is uniformly continuous in x, uniformly in t and right-continuous in t.*

Proof. The existence and uniqueness for the 2BSDE follows directly from Theorem 5.1. Since $\xi \in \mathrm{UC_b}(\Omega)$, we have with the notations of the previous section $V_t = u(t, B_t)$. But, by Proposition 5.2, we know that $Y_t = V_t$, hence the first result.

Then the uniform continuity of u is a simple consequence of Lemma 5.1. Finally, the right-continuity of u in t can be obtained exactly as in the proof of Theorem 5.9 in [24]. □

7.1 Non-linear Feynman-Kac formula in the quadratic case

Exactly as in the classical case and as in Theorem 5.3 in [24], we have a non-linear version of the Feynaman-Kac formula. The proof is the same as in [24], so we omit it. Notice however that it is more involved than in the classical case, mainly due to the technicalities introduced by the quasi-sure framework.

Theorem 7.2. *Under Assumption 7.1, suppose that $\widehat{h}$ is continuous in its domain, that D_f is independent of t and is bounded both from above and away from 0. Let $v \in C^{1,2}([0,T),\mathbb{R}^d)$ be a classical solution of (7.4) with $\{(v,Dv)(t,B_t)\}_{0\leq t\leq T} \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H$. Then*

$$Y_t := v(t,B_t),\ Z_t := Dv(t,B_t),\ K_t := \int_0^t k_s ds,$$

is the unique solution of the quadratic 2BSDE (7.3), where

$$k_t := \widehat{h}(t,B_t,Y_t,Z_t,\Gamma_t) - \frac{1}{2}\mathrm{Tr}\left[\widehat{a}_t^{1/2}\Gamma_t\right] + f(t,B_t,Y_t,Z_t,\widehat{a}_t) \text{ and } \Gamma_t := D^2 v(t,B_t).$$

7.2 The viscosity solution property

As usual when dealing with possibly discontinuous viscosity solutions, we introduce the following upper and lower-semicontinuous envelopes

$$u_*(t,x) := \underline{\lim}_{(t',x')\to(t,x)} u(t',x'),\ u^*(t,x) := \overline{\lim}_{(t',x')\to(t,x)} u(t',x')$$
$$\widehat{h}_*(\vartheta) := \underline{\lim}_{(\vartheta')\to(\vartheta)} \widehat{h}(\vartheta'),\ \widehat{h}^*(\vartheta) := \overline{\lim}_{(\vartheta')\to(\vartheta)} \widehat{h}(\vartheta')$$

In order to prove the main Theorem of this subsection, we will need the following Proposition, whose proof (which is rather technical) is omitted, since it is exactly the same as the proof of Propositions 5.10 and 5.14 and Lemma 6.2 in [24].

Proposition 7.1. *Let Assumption 7.1 hold. Then for any bounded function g and (t,x)*

(i) $\forall \left\{\tau^{\mathbb{P}},\mathbb{P}\in\mathcal{P}_h^t\right\}$, *$\mathbb{F}^t$-stopping times , we have* $u(t,x) \leq \sup_{\mathbb{P}\in\mathcal{P}_h^t} \mathcal{Y}_t^{\mathbb{P},t,x}(\tau^{\mathbb{P}}, u^*(\tau^{\mathbb{P}}, B^{t,x}_{\tau^{\mathbb{P}}}))$.

(ii) *If in addition g is lower-semicontinuous, then* $u(t,x) = \sup_{\mathbb{P}\in\mathcal{P}_h^t} \mathcal{Y}_t^{\mathbb{P},t,x}(\tau^{\mathbb{P}}, u(\tau^{\mathbb{P}}, B^{t,x}_{\tau^{\mathbb{P}}}))$.

Now we can state the main Theorem of this section

Theorem 7.3. *Let Assumption 7.1 hold true. Then*

(i) *u is a viscosity subsolution of*

$$-\partial_t u^* - \widehat{h}^*(\cdot,u^*,Du^*,D^2u^*) \leq 0, \text{ on } [0,T)\times\mathbb{R}^d.$$

(ii) *If in addition g is lower-semicontinuous and D_f is independent of t, then u is a viscosity supersolution of*

$$-\partial_t u_* - \widehat{h}_*(\cdot,u_*,Du_*,D^2u_*) \geq 0, \text{ on } [0,T)\times\mathbb{R}^d.$$

Proof. The proof follows closely the proof of Theorem 5.11 in [24], with some minor modifications (notably when we prove (7.10)). We provide the proof of the supersolution property for the convenience of the reader, the subsolution property can be proved similarly. Assume to the contrary that for some $(t_0, x_0) \in [0,T) \times \mathbb{R}^d$,

$$0 = (u^* - \phi)(t_0, x_0) > (u^* - \phi)(t,x) \text{ for all } (t,x) \in [0,T) \times \mathbb{R}^d \backslash \{(t_0,x_0)\}, \tag{7.7}$$

$$\left(-\partial_t \phi - \widehat{h}^*(\cdot, \phi, D\phi, D^2\phi)\right)(t_0, x_0) > 0, \tag{7.8}$$

for some smooth and bounded function ϕ (we can assume w.l.o.g. that ϕ is bounded since we are working with bounded solutions of 2BSDEs). Now since ϕ is smooth and since by definition $\widehat{h}^*$ is upper-semicontinuous, there exists an open ball $\mathcal{O}(r, (t_0, x_0))$ centered at (t_0, x_0) with radius r, which can be chosen less than $T - t_0$, such that

$$-\partial_t \phi - \widehat{h}(\cdot, \phi, D\phi, D^2\phi) \geq 0, \text{ on } \mathcal{O}(r, (t_0, x_0)).$$

By definition of $\widehat{h}$, this implies that for any $\alpha \in \mathbb{S}_d^{>0}$

$$-\partial_t \phi - \frac{1}{2}\text{Tr}\left[\alpha D^2\phi\right] + f(\cdot, \phi, D\phi, \alpha) \geq 0, \text{ on } \mathcal{O}(r, (t_0, x_0)). \tag{7.9}$$

Let us now denote $\mu := -\max_{\partial\mathcal{O}(r,(t_0,x_0))}(u^* - \phi)$. By (7.7), this quantity is strictly positive. Let now (t_n, x_n) be a sequence in $\mathcal{O}(r, (t_0, x_0))$ such that $(t_n, x_n) \to (t_0, x_0)$ and $u(t_n, x_n) \to u^*(t_0, x_0)$. Denote the following stopping time

$$\tau_n := \inf\left\{s > t_n,\ (s, B_s^{t_n,x_n} \notin \mathcal{O}(r, (t_0, x_0))\right\}.$$

Since $r < T - t_0$, we have $\tau_n < T$ and thus $(\tau_n, B_{\tau_n}^{t_n,x_n}) \in \partial\mathcal{O}(r, (t_0, x_0))$. Hence, we have

$$c_n := (\phi - u)(t_n, x_n) \to 0 \text{ and } u^*(\tau_n, B_{\tau_n}^{t_n,x_n}) \leq \phi(\tau_n, B_{\tau_n}^{t_n,x_n}) - \mu.$$

Fix now some $\mathbb{P}^n \in \mathcal{P}_h^{t_n}$. By the comparison Theorem for quadratic BSDEs, we have

$$\mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n, u^*(\tau_n, B_{\tau_n}^{t_n,x_n})) \leq \mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n, \phi(\tau_n, B_{\tau_n}^{t_n,x_n}) - \mu).$$

Then proceeding exactly as in the second step of the proof of Theorem 3.1, we can define a bounded process M_n, whose bounds only depend on T and the Lipchitz constant of f in y, and a probability measure $\mathbb{Q}_n$ equivalent to $\mathbb{P}_n$ such that

$$\mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n, \phi(\tau_n, B_{\tau_n}^{t_n,x_n}) - \mu) - \mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n, \phi(\tau_n, B_{\tau_n}^{t_n,x_n})) = -\mathbb{E}_{t_n}^{\mathbb{Q}_n}[M_{\tau_n}\mu] \leq -\mu',$$

for some strictly positive constant μ' which is independent of n. Hence, we obtain by definition of c_n

$$\mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n, u^*(\tau_n, B_{\tau_n}^{t_n,x_n})) - u(t_n, x_n) \leq \mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n, \phi(\tau_n, B_{\tau_n}^{t_n,x_n})) - \phi(t_n, x_n) + c_n - \mu'. \tag{7.10}$$

With the same arguments as above, it is then easy to show with Itô's formula that

$$\mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n, \phi(\tau_n, B_{\tau_n}^{t_n,x_n})) - \phi(t_n, x_n) = \mathbb{E}_{t_n}^{\mathbb{Q}_n}\left[-\int_{t_n}^{\tau_n} M_s^n \psi_s^n ds\right],$$

where $\psi_s^n := (-\partial_t \phi - \frac{1}{2}\mathrm{Tr}\left[\widehat{a}_s^t D^2\phi\right] + f(\cdot, D\phi, \widehat{a}_s^t))(s, B_s^{t_n,x_n})$. But by (7.9) and the definition of τ_n, we know that for $t_n \leq s \leq \tau_n$, $\psi_s^n \geq 0$. Recalling (7.10), we then get

$$\mathcal{Y}_{t_n}^{\mathbb{P}_n,t_n,x_n}(\tau_n, u^*(\tau_n, B_{\tau_n}^{t_n,x_n})) - u(t_n, x_n) \leq c_n - \mu'.$$

Since c_n does not depend on $\mathbb{P}_n$, and since the right-hand side is strictly negative for n large enough, this contradicts Proposition 7.1(i). □

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